MECA-H303 Vibration TP3

Lagrangian dynamics

Double pendulum

For the following system:

1. Derive the equations of motion of the system using Lagrangian dynamics.
2. Linearise the system and write the equations of motion in matrix form.
3. Write the expression of the resonant frequencies and eigen modes of the system.

Additional details for the exercise:

- Assume small angles of rotation and neglect second order terms for te linearisation.
- $\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) = \cos(\alpha - \beta)$
- The weight of the bars is negligeable
- Each black disc has a mass m, an inertia I and is fixed to the bar above it
- The final answer should be expressed as a function of the coordinates on the schematic

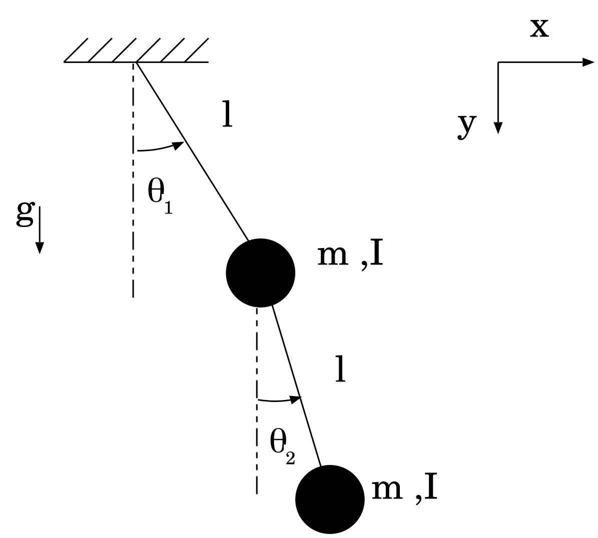

Figure 1: Double Pendulum with discs

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$$T = \frac{1}{2} m v_1^2 + \frac{1}{2} I \dot{\theta}_1^2 + \frac{1}{2} m v_2^2 + \frac{1}{2} I \dot{\theta}_2^2$$

$$v_1 = \frac{d}{dt}\left(l \sin\theta_1 \overline{1}_x - l \cos\theta_1 \overline{1}_y\right) = l\dot{\theta}_1 \cos\theta_1 \overline{1}_x + l\dot{\theta}_1 \sin\theta_1 \overline{1}_y = l\dot{\theta}_1 \overline{1}_{\theta_1}$$

$$v_2 = l\frac{d}{dt}\left(\left(\sin\theta_1 + \sin\theta_2\right)\overline{1}_x - \left(\cos\theta_1 + \cos\theta_2\right)\overline{1}_y\right)$$

$$= l\left(\dot{\theta}_1 \cos\theta_1 + \dot{\theta}_2 \cos\theta_2\right)\overline{1}_x + l\left(\dot{\theta}_1 \sin\theta_1 + \dot{\theta}_2 \sin\theta_2\right)\overline{1}_y$$

$$v_2^2 = l^2\Big(\underbrace{\dot{\theta}_1^2 \cos^2\theta_1}_{\mapsto = \dot{\theta}_1^2} + \underbrace{\dot{\theta}_2^2 \cos^2\theta_2}_{\mapsto = \dot{\theta}_2^2} + \underbrace{2\dot{\theta}_1\dot{\theta}_2 \cos\theta_1 \cos\theta_2}_{\mapsto = 2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) = 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)}$$

$$+ \dot{\theta}_1^2 \sin^2\theta_1 + \dot{\theta}_2^2 \sin^2\theta_2 + 2\dot{\theta}_1\dot{\theta}_2 \sin\theta_1 \sin\theta_2\Big)$$

$$= \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) \quad = \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)$$

Blue colour will represent the other possible solution.

$$T = \frac{1}{2} m l^2 \dot{\theta}_1^2 + \frac{1}{2} I\dot{\theta}_1^2 + \frac{1}{2} m l^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right) + \frac{1}{2} I \dot{\theta}_2^2$$

$$V = -mgl\cos\theta_1 - mgl\left(\cos\theta_1 + \cos\theta_2\right)$$

not $\frac{1}{2} I\left(\dot{\theta}_2 + \dot{\theta}_1\right)^2$ because we use absolute coordinates

$$L = T - V$$

$$= ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\left(\dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right) + \frac{1}{2}I\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right)$$

$$+ 2mgl\cos\theta_1 + mgl\cos\theta_2$$

$$\frac{d}{dt}\frac{dL}{d\dot{q}_i} - \frac{dL}{dq_i} = Q_i \quad (Q_i = 0)$$

<u>variable θ_1</u>

$$\frac{dL}{d\theta_1} = m l^2 \dot{\theta}_1 \dot{\theta}_2 (-\sin(\theta_1 - \theta_2)) - 2 mgl \sin\theta_1$$

$$(+\sin(\theta_2 - \theta_1))$$

$$\frac{dL}{d\dot{\theta}_1} = m l^2 \dot{\theta}_1 + I \dot{\theta}_1 + m l^2 \dot{\theta}_1 + m l^2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$2 m l^2 \ddot{\theta}_1 + m l^2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m l^2 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \cdot (\dot{\theta}_1 - \dot{\theta}_2)$$

$$+ I \ddot{\theta}_1 + m l^2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + 2 mgl \sin\theta_1 = 0$$

$$- m l^2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1)$$

$$\Leftrightarrow (2 m l^2 + I) \ddot{\theta}_1 + m l^2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) \pm m l^2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)$$

$$+ 2 mgl \sin\theta_1 = 0$$

variable θ_2

$$\frac{dL}{d\theta_2} = m l^2 \dot\theta_1 \dot\theta_2 \left(+\sin(\theta_1 - \theta_2)\right) - m g l \sin\theta_2$$

$$\left(-\sin(\theta_2 - \theta_1)\right)$$

$$\frac{dL}{d\dot\theta_2} = m l^2 \dot\theta_2 + m l^2 \dot\theta_1 \cos(\theta_1 - \theta_2) + I \dot\theta_2$$

$$m l^2 \ddot\theta_2 + m l^2 \ddot\theta_1 \cos(\theta_1 - \theta_2) - m l^2 \dot\theta_1 \sin(\theta_1 - \theta_2) \cdot (\dot\theta_1 - \dot\theta_2)$$

$$+ I \ddot\theta_2 - m l^2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) + m g l \sin\theta_2 = 0$$

$+$

$$\Leftrightarrow m l^2 \ddot\theta_1 \cos(\theta_1 - \theta_2) + (m l^2 + I)\ddot\theta_2 + m g l \sin\theta_2$$

$$- m l^2 \dot\theta_1^2 \sin(\theta_1 - \theta_2) = 0$$

$+$

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θ_1 and θ_2 are small

$\sin(\theta_1) \approx \theta_1 \quad \cos(\theta_1) \approx 1$

neglecting 2^{nd} order terms

$\dot{\theta}_1 \cdot \dot{\theta}_2 \approx 0 \quad \dot{\theta}^2 \approx 0$

$$\begin{bmatrix} 2ml^2+I & ml^2 \\ ml^2 & ml^2+I \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2mgl & 0 \\ 0 & mgl \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$M \ddot{\Theta} + K \dot{\Theta} = 0$$

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<u>natural frequencies / eigen values</u>

$$\det(K - \omega^2 M) = 0 \iff \omega_i^2 = ?$$

$$\iff \det\left(\begin{bmatrix} 2ml^2+I & ml^2 \\ ml^2 & ml^2+I \end{bmatrix} - \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \begin{bmatrix} 2mgl & 0 \\ 0 & mgl \end{bmatrix} \right) = 0$$

$$\iff \det\left(\begin{bmatrix} 2ml^2+I & ml^2 \\ ml^2 & ml^2+I \end{bmatrix} - \begin{bmatrix} 2mgl\omega^2 & 0 \\ 0 & mgl\omega^2 \end{bmatrix} \right) = 0$$

$$\Leftrightarrow \det\left(\begin{bmatrix} 2ml^2+I-2mgl\omega^2 & ml^2 \\ ml^2 & ml^2+I-mgl\omega^2 \end{bmatrix}\right) = 0$$

$$\Leftrightarrow \left((2ml^2+I)-2mgl\omega^2\right)\left((ml^2+I)-mgl\omega^2\right) - m^2 l^4 = 0$$

$$\Leftrightarrow (2ml^2+I)(ml^2+I) - (2ml^2+I)mgl\omega^2 - (ml^2+I)2mgl\omega^2 \ldots$$

$$\ldots + 2m^2g^2l^2\omega^4 - m^2l^4 = 0$$

$$\Leftrightarrow 2m^2g^2l^2\omega^4 - mgl\omega^2(4m^2+3I) - m^2l^4 + (2ml^2+I)(ml^2+I) = 0$$

assume $\omega^2 = \lambda$

$$\Leftrightarrow 2m^2g^2l^2\lambda^2 - (4m^2+3I)mgl\lambda - m^2l^4 + \underbrace{(2ml^2+I)(ml^2+I)}_{=2m^2l^4+3Iml^2+I^2} = 0$$

$$\Delta = (4m^2+3I)^2m^2g^2l^2 - 4 \cdot 2m^2g^2l^2 \cdot (m^2l^4+3Iml^2+I^2)$$

$$\lambda_{1,2} = \frac{(4m^2+3I)mgl \pm \sqrt{\Delta}}{4m^2g^2l^2} \quad \begin{cases} \lambda_1 = \omega_1^2 = \ldots \\ \lambda_2 = \omega_2^2 = \ldots \end{cases}$$

$$\rightarrow \frac{(4m^2+3I) \pm \sqrt{(4m^2+3I)^2 - 8(m^2l^4+3Iml^2+I^2)}}{4mgl}$$

mode shapes (eigen vectors)

$$(K-\omega^2 M)\Psi = \{0\} \Leftrightarrow \Psi_i = ?$$

$$\begin{bmatrix} 2ml^2+I-2mgl\omega^2 & ml^2 \\ ml^2 & ml^2+I-mgl\omega^2 \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} (2ml^2+I-2mgl\omega^2)\varphi_1 + ml^2\varphi_2 = 0 \\ ml^2\varphi_1 + (ml^2+I-mgl\omega^2)\varphi_2 = 0 \end{cases}$$

$$\begin{cases} \varphi_1 = -\varphi_2 \dfrac{ml^2}{2ml^2+I-2mgl\omega^2} \\ \varphi_1 = -\varphi_2 \dfrac{ml^2+I-mgl\omega^2}{ml^2} \end{cases}$$

Identity relationship

$$\frac{ml^2}{2ml^2+I-2mgl\omega^2} = \frac{ml^2+I-mgl\omega^2}{ml^2}$$

$$\boxed{\omega^2 = \lambda_1}$$

we choose $\varphi_2 = 1 \rightarrow \varphi_1 = \dfrac{-ml^2}{2ml^2+I-2mgl\lambda_1}$

$$\Psi_1 = \begin{Bmatrix} \dfrac{-ml^2}{2ml^2+I-2mgl\lambda_2} \\ 1 \end{Bmatrix}$$

$$\boxed{\omega^2 = \lambda_2}$$

we choose $\varphi_2 = 1 \rightarrow \varphi_1 = \dfrac{-ml^2}{2ml^2+I-2mgl\lambda_2}$

$$\Psi_2 = \begin{Bmatrix} \dfrac{-ml^2}{2ml^2+I-2mgl\lambda_2} \\ 1 \end{Bmatrix}$$