One DOF systems


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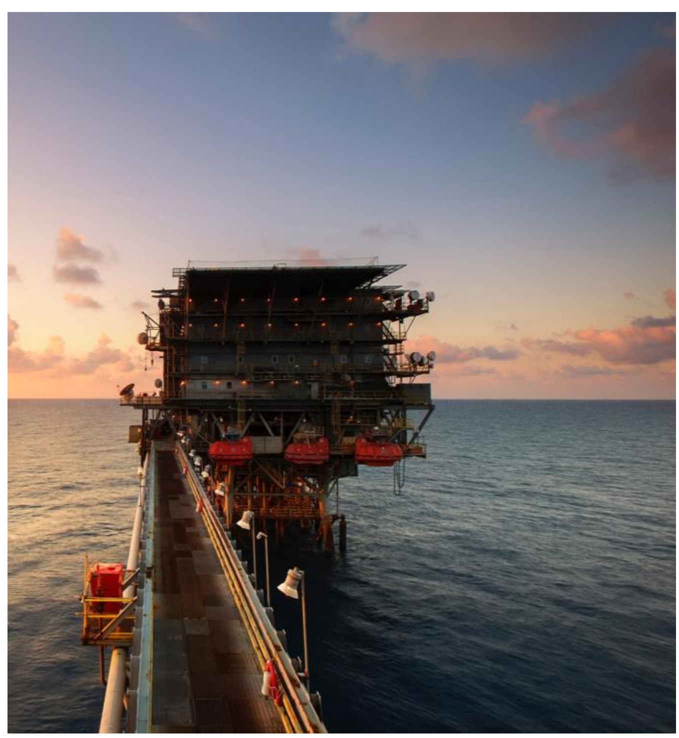
ONE DEGREE OF FREEDOM SYSTEMS IN REAL LIFE



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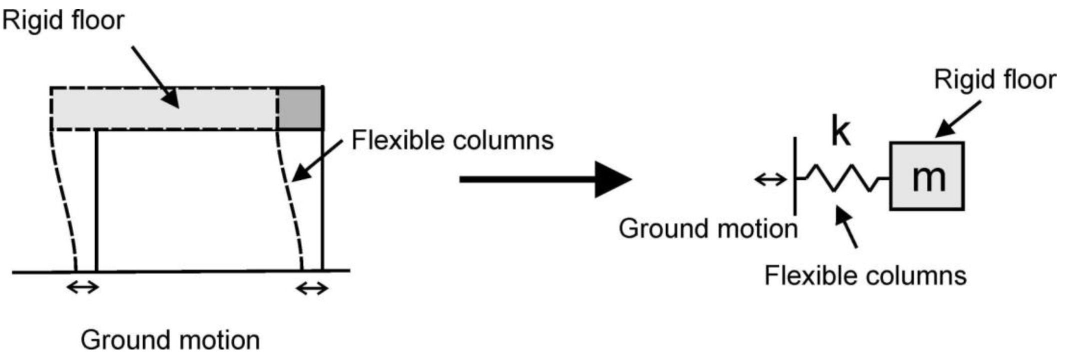
Reduction of a system to a one DOF system

Example 1:



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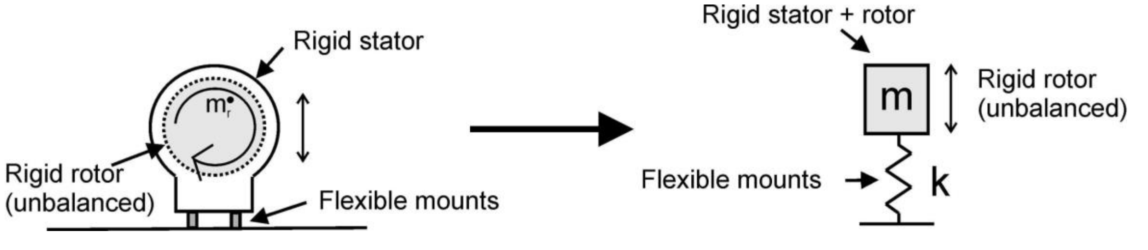
Reduction of a system to a one DOF system

Example 2:



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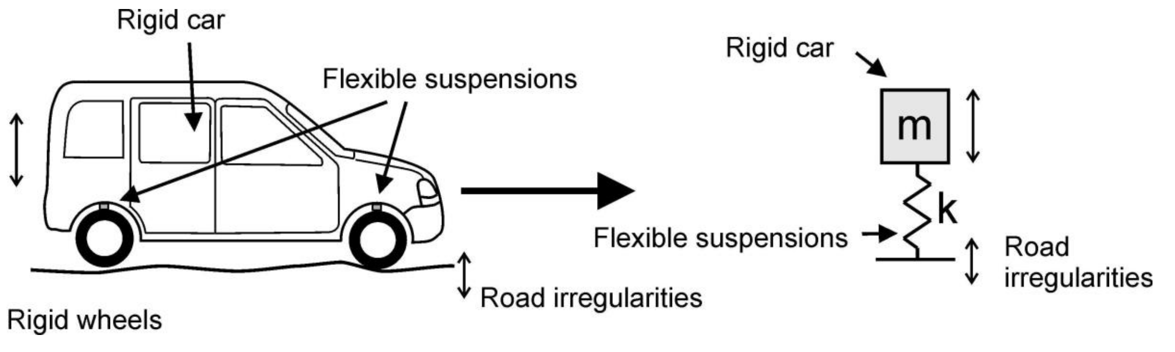
Reduction of a system to a one DOF system

Example 3:



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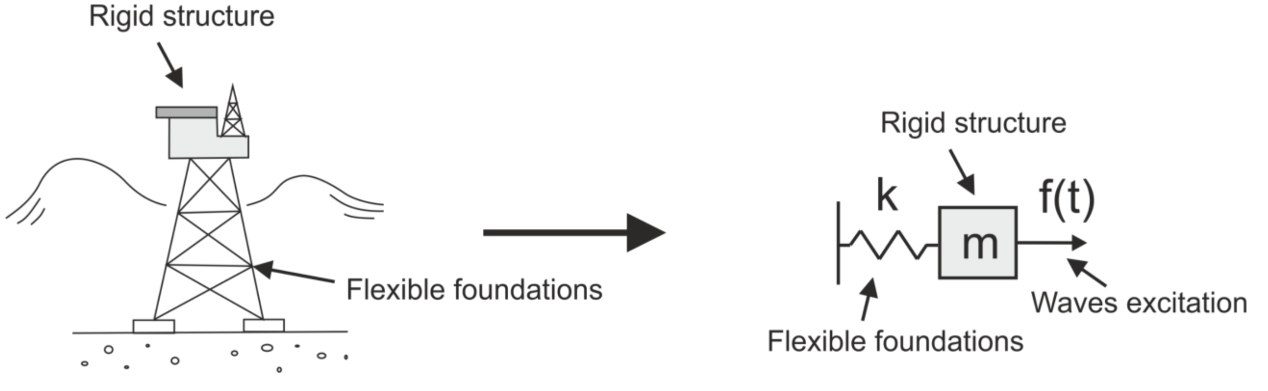
Reduction of a system to a one DOF system

Example 4:



Offshore platform

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HARMONIC MOTION

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Simple harmonic motion

https://www.youtube.com/watch?v=gZ_KnZHCn4M

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Harmonic signals

A periodic vibration of which the amplitude can be described by a sinusoidal function:

$$u(t) = a\cos(\omega t + \phi)$$
$$u(t) = a\sin(\omega t + \phi)$$

is called an harmonic vibration
with:

- amplitude a
- angular frequency $\omega = 2\pi f$
- frequency f
- period T = 1/f or f = 1/T
- phase angle ϕ at t=0
- total phase angle $\omega t + \phi$

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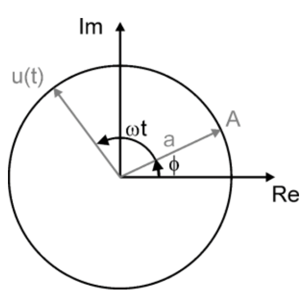
Harmonic signals

Representation in the complex plane:

$$\begin{aligned} u(t) &= a e^{i(\omega t + \phi)} \\ &= a\cos(\omega t + \phi) + ia\sin(\omega t + \phi) \end{aligned}$$

$$\begin{aligned} u(t) &= a e^{i\phi} e^{i\omega t} = A e^{i\omega t} \\ A &= \underline{a\cos\phi + ia\sin\phi} \end{aligned}$$

Independent of time



Projection of the rotating vector on the real axis is a cosine

Projection of the rotating vector on the imaginary axis is a sine

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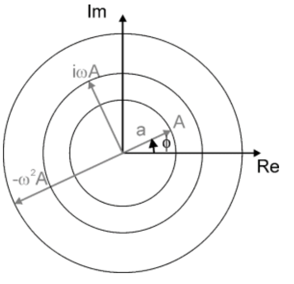
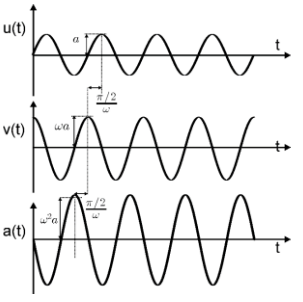
Harmonic signals

$$
\begin{aligned}
\text{Displacement} \quad u(t) &= Ae^{i\omega t} \\
\text{Velocity} \quad v(t) &= \frac{d\,u(t)}{dt} = i\omega Ae^{i\omega t} = i\omega u(t) \\
\text{Acceleration} \quad a(t) &= \frac{d\,v(t)}{dt} = -\omega^2 Ae^{i\omega t} = -\omega^2 u(t)
\end{aligned}
$$





the phase angle of u(t) is 90° behind v(t)

the phase angle of v(t) is 90° behind a(t)

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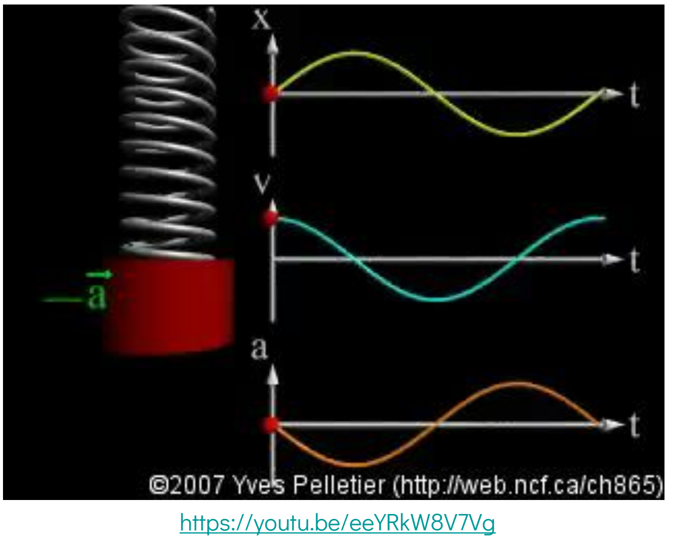
Harmonic signals



https://youtu.be/eeYRkW8V7Vg

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UNDAMPED VIBRATIONS OF A ONE DOF SYSTEM

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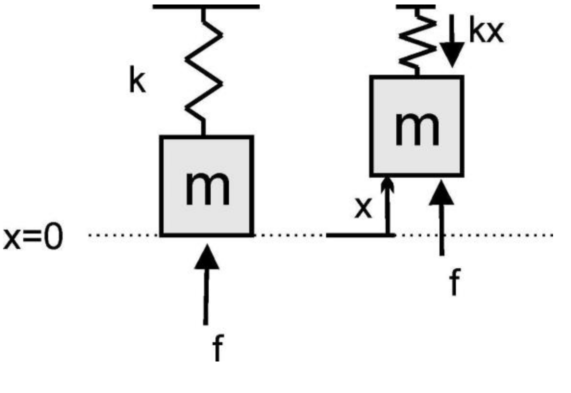
Equation of motion

Newton's law:

$$m\ddot{x} = \sum F_x$$

- Spring force: $-kx$
- External force f acting on the mass.

$$\boxed{m\ddot{x} + kx = f}$$



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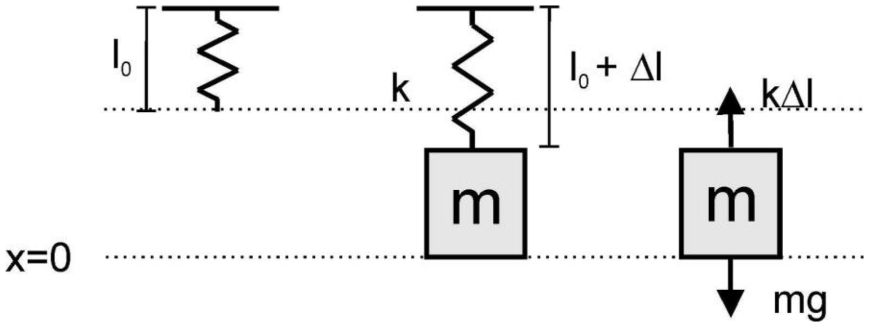
Equation of motion

What about the effect of gravity ?



The displacement x is defined with respect to the equilibrium position of the mass subjected to gravity. The effect of gravity should therefore not be taken into account in the equation of motion of the system.

Free vibrations

$$\boxed{m\ddot{x} + kx = 0} \qquad x = A\,e^{rt}$$

Characteristic equation:

$$mr^2 + k = 0 \qquad r = \pm i\sqrt{k/m}$$

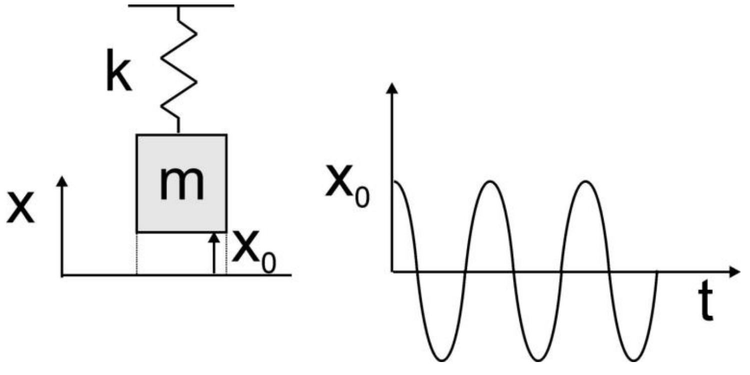
$$\boxed{x = A\cos\omega_n t + B\sin\omega_n t} \qquad \omega_n = \sqrt{k/m}$$

•In the absence of external excitation force, the motion is oscillatory. The natural angular frequency ω_n is defined by the values of k and m
•The motion is initialized by imposing initial conditions on the displacement and the velocity

Free vibrations

displacement x_0 velocity $\dot{x}_0$

$$\boxed{x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t}$$



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Free vibrations

$$\boxed{x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t}$$

Alternative representation:

$$x(t) = a \cos(\omega_n t + \phi)$$

$$\begin{aligned} x_0 &= a \cos \phi \\ \dot{x}_0/\omega_n &= -a \sin \phi \end{aligned} \qquad \tan \phi = -\frac{\dot{x}_0}{\omega_n x_0}$$

The motion can be described by a cosine function with a phase. The phase is a function of the initial conditions.

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Harmonic excitation

$$f(t) = F\,e^{i\omega t} \qquad x(t) = X\,e^{i\omega t}$$

$$m\ddot{x} + kx = f$$

$$\downarrow$$

$$\left(k - \omega^2 m\right) X e^{i\omega t} = F e^{i\omega t}$$

$$\downarrow$$

$$\boxed{X = \frac{F}{k - \omega^2 m}}$$

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Harmonic excitation

$$X = \frac{F}{k - \omega^2 m}$$

$$X_0 = F/k \quad (\omega = 0)$$

$$\omega_n = \sqrt{k/m}$$

$$\boxed{\frac{X}{X_0} = \frac{1}{1 - \omega^2/\omega_n^2}}$$

- Positive if $\omega < \omega_n$
- Infinite if $\omega = \omega_n$
- Negative if $\omega > \omega_n$

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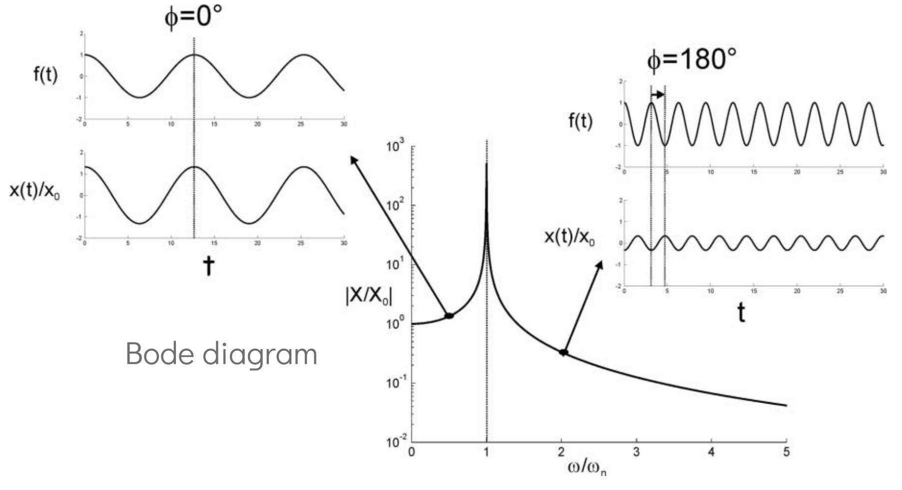
Harmonic excitation

$$\boxed{\frac{X}{X_0} = \frac{1}{1 - \omega^2/\omega_n^2}}$$

Dynamic amplification factor



Bode diagram

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RESONANCE

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Breaking a glass of wine with sound

https://www.youtube.com/watch?v=10lWpHyN0Ok

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Breaking a glass of wine with sound

https://www.youtube.com/watch?v=JiM6AtNLXX4

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Is stiffer stronger ?

https://youtu.be/n9ULMIjvSIg

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Is stiffer stronger ?

https://youtu.be/LV_UuzEznHs

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Buildings resonance

https://youtu.be/pMr1MzSv044

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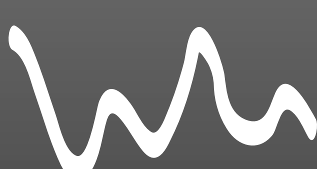
DAMPED VIBRATIONS OF A ONE DOF SYSTEM



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Effect of damping on a building

https://youtu.be/HWpkaIB1fD0

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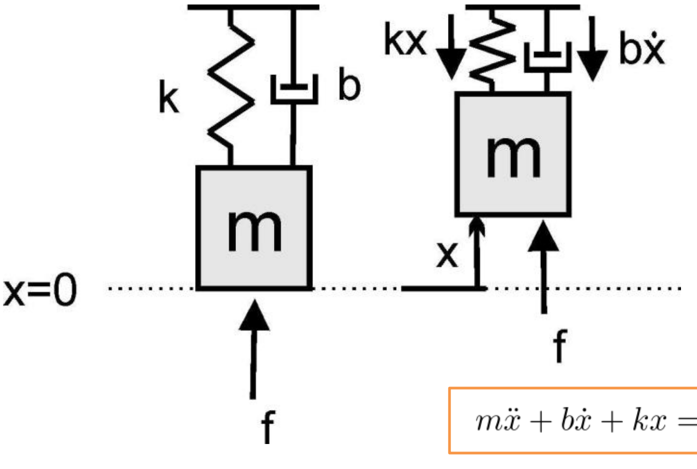
Equation of motion

Damping force : $F_b = -b\dot{x}$ viscous damping



$$\boxed{m\ddot{x} + b\dot{x} + kx = f}$$

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Free vibrations

$$m\ddot{x} + b\dot{x} + kx = f$$

$$\xi = \frac{b}{2\sqrt{km}}$$ damping coefficient

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = f/m$$

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0$$

General solution $x = A\,e^{rt}$

Characteristic equation:

$$r^2 + 2\xi\omega_n r + \omega_n^2 = 0$$

$$\xi < 1$$

$$w_d = \omega_n\sqrt{1-\xi^2}$$

$$r = -\xi\omega_n \pm i\omega_n\sqrt{1-\xi^2} = -\xi\omega_n \pm i\omega_d$$

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Free vibrations

$$x(t) = e^{-\xi\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

Initial conditions: displacement x_0 velocity $\dot{x_0}$

$$\boxed{x(t) = e^{-\xi\omega_n t}\left(x_0\cos\omega_d t + \frac{\dot{x_0} + \omega_n\xi x_0}{\omega_d}\sin\omega_d t\right)}$$

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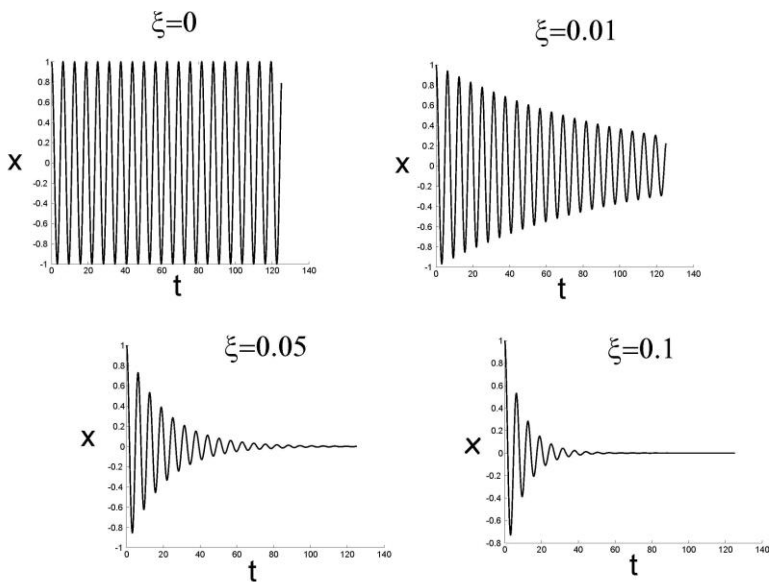
Free vibrations



Free response to unit initial displacement

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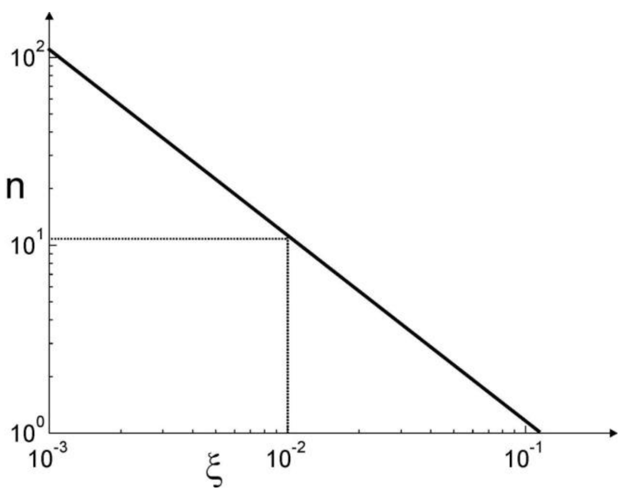
Free vibrations



Number of oscillations after which the vibration amplitude is reduced by one half

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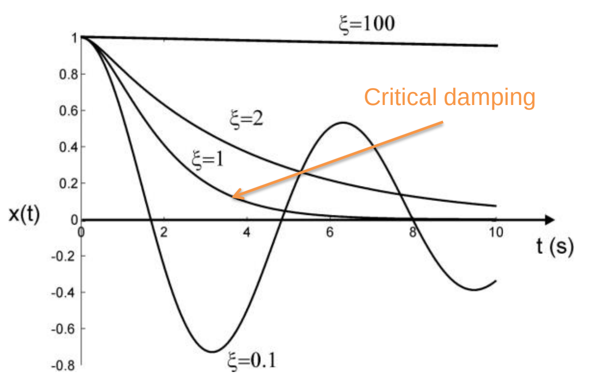
Free vibrations

$$\xi > 1 \qquad x(t) = e^{-\xi\omega_n t}\left(x_0 \cosh \mu t + \frac{\dot{x}_0 + \omega_n \xi x_0}{\mu} \sinh \mu t\right)$$

$$\mu = \omega_n \sqrt{\xi^2 - 1}$$

$$\xi = 1 \qquad x(t) = e^{-\omega_n t}\left((\dot{x}_0 + \omega_n x_0)\, t + x_0\right)$$

Critical damping



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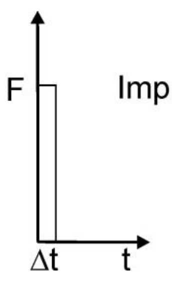
Impulse response



Impulse=FΔt

$$m\ddot{x} + b\dot{x} + kx = f \qquad x_0 = 0,\ \dot{x}_0 = 0$$

$$m\dot{x}_0|_{\Delta t} = F\Delta t - \cancel{\int_0^{\Delta t} kx\,dt} - \cancel{\int_0^{\Delta t} b\dot{x}\,dt}$$

$$\boxed{\dot{x}_0|_{\Delta t} = \frac{F\Delta t}{m}}$$

Equivalent to initial velocity at Δt

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Impulse response

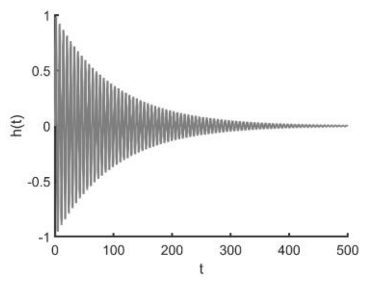
For an initial velocity, the response of the system is:

$$x(t) = \frac{e^{-\xi\omega_n t}\dot{x}_0}{\omega_d}\sin(\omega_d t)$$

with $\dot{x}_0 = \dfrac{F\Delta t}{m}$

For a unit impulse $F\Delta t = 1$, we define the impulse response *h(t)*

$$\boxed{h(t) = \frac{e^{-\xi\omega_n t}}{m\omega_d}\sin(\omega_d t)}$$



$\omega_n = 1, \xi = 0.01$

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Harmonic excitation

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = f/m$$

$$x(t) = X\,e^{i\omega t}$$
$$f(t) = F\,e^{i\omega t}$$

$$(\omega_n^2 + 2i\xi\omega\omega_n - \omega^2)X = F/m$$

$$X = \frac{F}{m}\left(\frac{1}{\omega_n^2 + 2i\xi\omega\omega_n - \omega^2}\right) = \frac{F}{k}\left(\frac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2i\xi\frac{\omega}{\omega_n}}\right)$$
$$= X_0\left(\frac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2i\xi\frac{\omega}{\omega_n}}\right)$$

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Harmonic excitation

$$X_r = X_0 \frac{1-\frac{\omega^2}{\omega_n^2}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}$$
$$X_i = X_0 \frac{-2\xi\frac{\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}$$

$$\begin{aligned} |X/X_0| &= \sqrt{\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \\ \tan\phi &= \frac{-2\xi\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}} \end{aligned}$$

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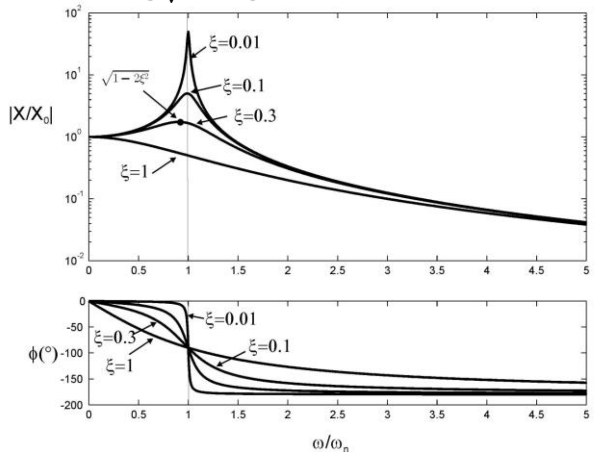
Harmonic excitation

Maximum $\omega/\omega_n = \sqrt{1-2\xi^2}$

Max ampl. $|X/X_0| = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$



For ξ small

$$\begin{aligned} \omega/\omega_n &= 1 \\ |X/X_0| &= \frac{1}{2\xi} \end{aligned}$$

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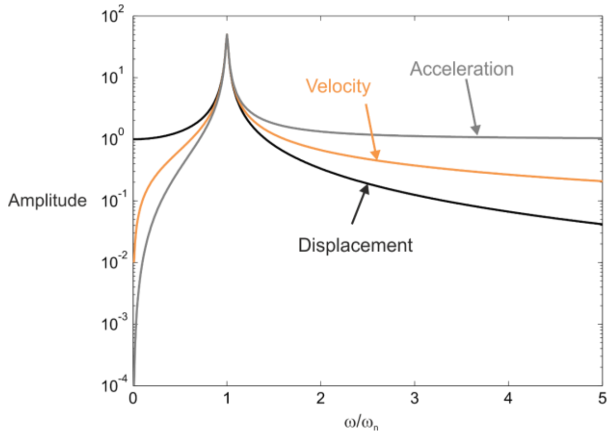
Harmonic excitation

$$
\begin{aligned}
\text{Displacement} \quad u(t) &= Ae^{i\omega t} \\
\text{Velocity} \quad v(t) &= \frac{d\,u(t)}{dt} = i\omega A e^{i\omega t} = i\omega u(t) \\
\text{Acceleration} \quad a(t) &= \frac{d\,v(t)}{dt} = -\omega^2 A e^{i\omega t} = -\omega^2 u(t)
\end{aligned}
$$



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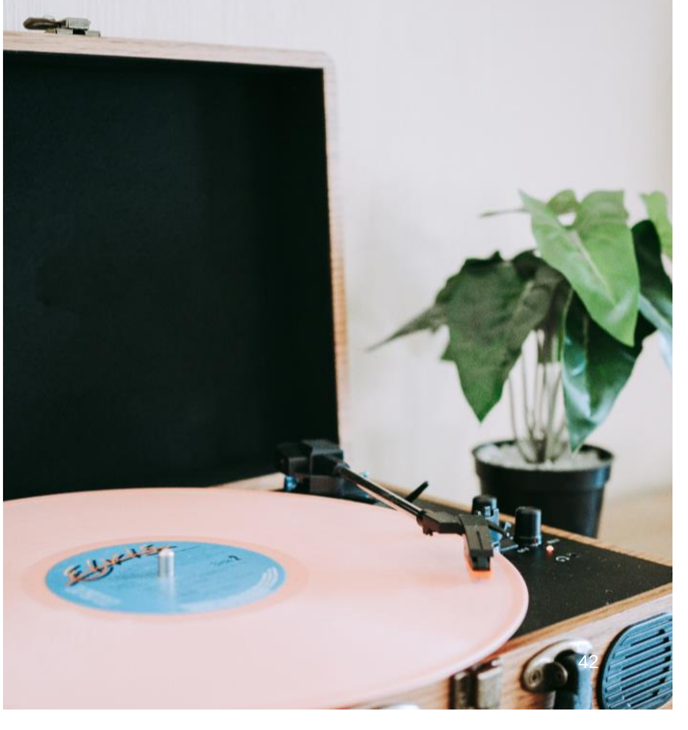
DYNAMIC RESPONSE OF ROTATING MACHINES



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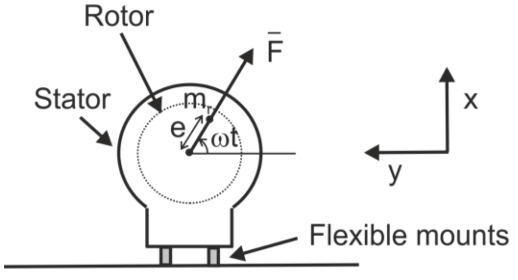
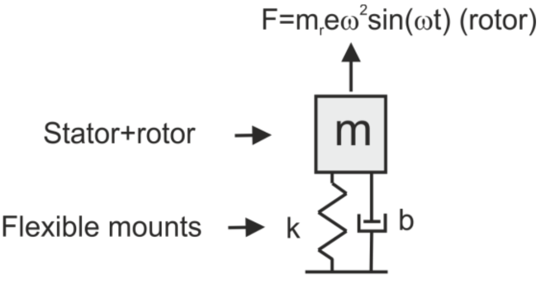
Rotating maching excitation

$$F = m_r e \omega^2 \sin(\omega t)$$





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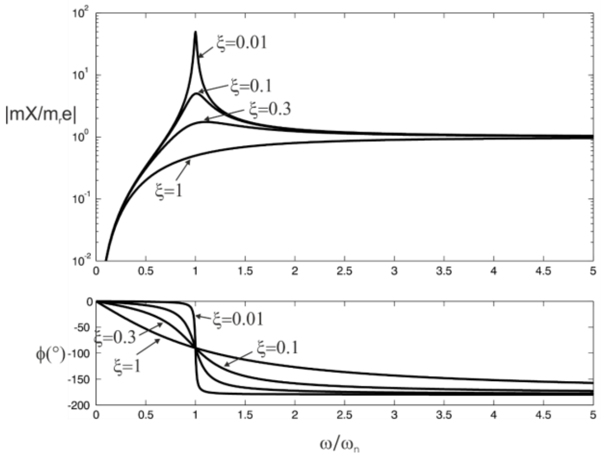
Rotating maching response

$$X = \frac{m_r e \omega^2}{k}\left(\frac{1}{1-\frac{\omega^2}{\omega_n^2}+2i\xi\frac{\omega}{\omega_n}}\right) \qquad \frac{mX}{m_r e} = \frac{\omega^2}{\omega_n^2}\left(\frac{1}{1-\frac{\omega^2}{\omega_n^2}+2i\xi\frac{\omega}{\omega_n}}\right)$$



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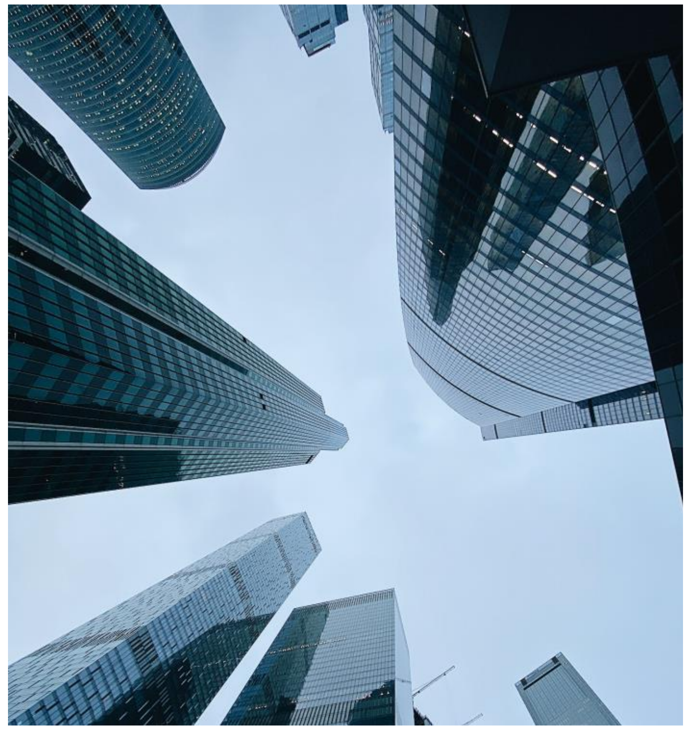
DYNAMIC RESPONSE TO BASE EXCITATION



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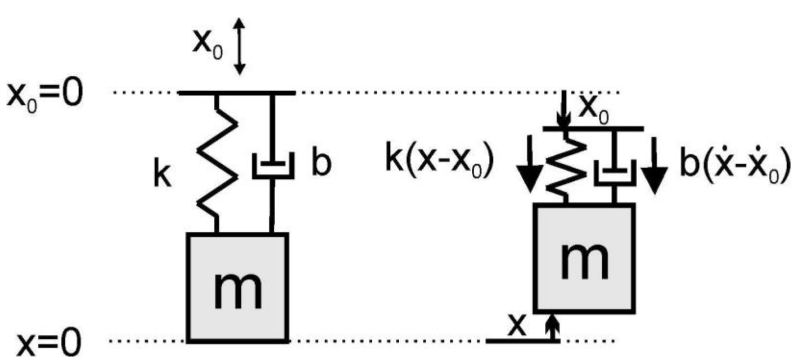
Equation of motion



$$m\ddot{x} = -k(x - x_0) - b(\dot{x} - \dot{x_0})$$

$$x_r = x - x_0$$

$$m\ddot{x_r} + b\dot{x_r} + kx_r = \underline{-m\ddot{x_0}}$$

Excitation

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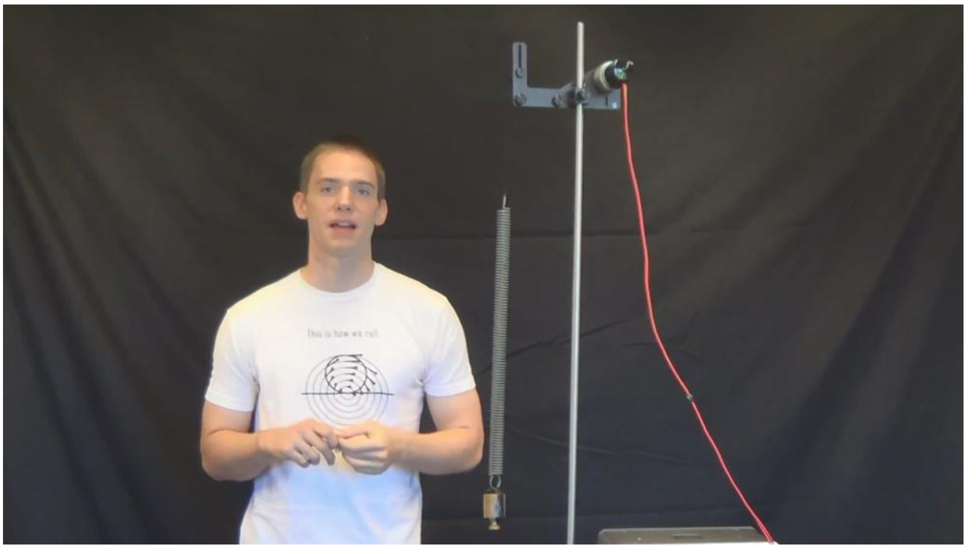
Harmonic response



https://youtu.be/cfKwnTfNhog

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