Lagrange equations

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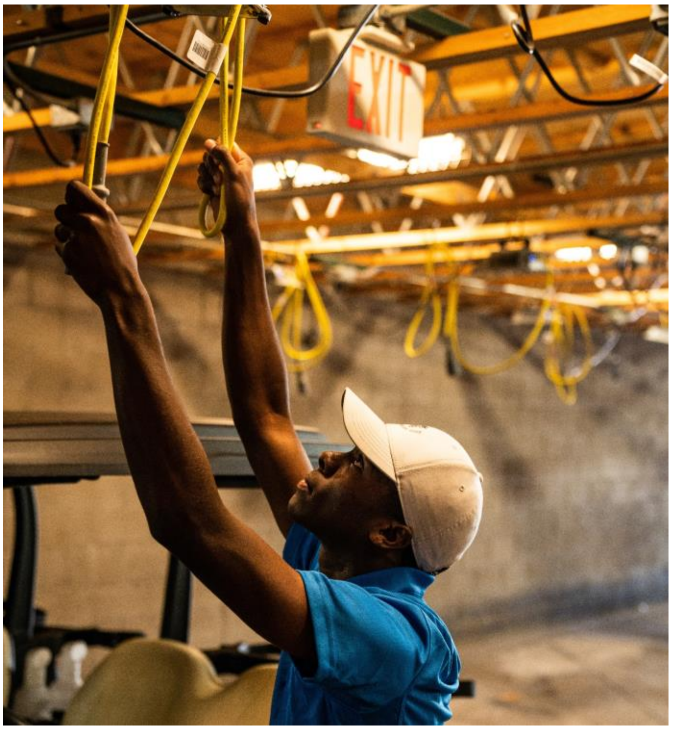
VIRTUAL DISPLACEMENTS and VIRTUAL WORK



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Principle of virtual work

Static equilibrium

$$\sum_{i=1}^{N} \overline{R}_i \cdot \overline{\delta x}_i = 0$$

$$\overline{R}_i = \overline{F}_i + \overline{F}_i'$$

Applied forces

Reaction forces

The virtual work of the constraint forces is zero for any virtual displacement

$$\overline{F_i'} \cdot \overline{\delta x}_i = 0$$

$$\sum_{i=1}^{N} \overline{F}_i \cdot \overline{\delta x}_i = 0$$

Virtual work of external forces is zero

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D'ALEMBERT AND HAMILTON's PRINCIPLES

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D'Alembert's principle

Extension to dynamics

$$\overline{R}_i = \overline{F}_i + \overline{F}'_i - m_i\ddot{\overline{x}}_i = 0 \longrightarrow \sum_{i=1}^{N} (\overline{F}_i - m_i\ddot{\overline{x}}_i) \cdot \overline{\delta x}_i = 0$$

If time does not appear explicitely in the constraints $\overline{\delta x}_i = \dot{\overline{x}}_i dt$

$$\sum_{i=1}^{N} \overline{F}_i \cdot \overline{\delta x}_i - \sum_{i=1}^{N} m_i\ddot{\overline{x}}_i \cdot \dot{\overline{x}}_i dt = 0$$

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D'Alembert's principle

If external forces can be expressed as the gradient of a potential V

$$\sum_{i=1}^{N} \overline{F}_i \cdot \overline{\delta x}_i = -dV$$

The second terms is the differential of the kinetic energy T :

$$m_i\ddot{\overline{x}}_i \cdot \dot{\overline{x}}_i dt = \frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^{N} m_i\,\dot{\overline{x}}_i \cdot \dot{\overline{x}}_i\right) dt = dT$$

Law of conservation of total energy:

$$d\,(T + V) = 0 \longrightarrow \boxed{T + V = Cst}$$

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Hamilton's principle

D'Alembert's principle cannot be formulated in general coordinates

$$\sum_{i=1}^{N} (\overline{F}_i - m_i \ddot{\overline{x}}_i) \cdot \overline{\delta x}_i = 0$$

Virtual work of applied external forces

$$\delta W = \sum_{i=1}^{N} \overline{F}_i \cdot \overline{\delta x}_i$$

$$\ddot{\overline{x}}_i \cdot \overline{\delta x}_i = \frac{d}{dt} \left(\dot{\overline{x}}_i \cdot \overline{\delta x}_i \right) - \delta \frac{1}{2} \left(\dot{\overline{x}}_i \cdot \dot{\overline{x}}_i \right)$$

$$\longrightarrow \sum_{i=1}^{N} m_i \ddot{\overline{x}}_i \cdot \overline{\delta x}_i = \sum_{i=1}^{N} m_i \frac{d}{dt} (\dot{\overline{x}}_i \cdot \overline{\delta x}_i) - \delta T$$

$$\delta W + \delta T = \sum_{i=1}^{N} m_i \frac{d}{dt} (\dot{\overline{x}}_i \cdot \overline{\delta x}_i)$$

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Hamilton's principle

$$\delta W + \delta T = \sum_{i=1}^{N} m_i \frac{d}{dt} (\dot{\overline{x}}_i \cdot \overline{\delta x}_i)$$

Integration over a time interval $[t_1, t_2]$ with $\overline{\delta x}_i(t_1) = \overline{\delta x}_i(t_2) = 0$

$$\longrightarrow \int_{t_1}^{t_2} (\delta W + \delta T)\, dt = 0$$

$$\delta W = -\delta V + \delta W_{nc}$$

Forces deriving from a potential ($-\delta V$)

Non-conservative forces (δW_{nc})

$$\longrightarrow \int_{t_1}^{t_2} (\delta L + \delta W_{nc})\, dt = 0$$

$$L = T - V$$

Lagrangian

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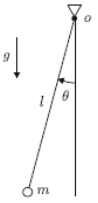
Example of Lagrangian



$$T = \frac{1}{2} m v_m^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$V = -mgy = -mgl \cos\theta$$

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos\theta$$

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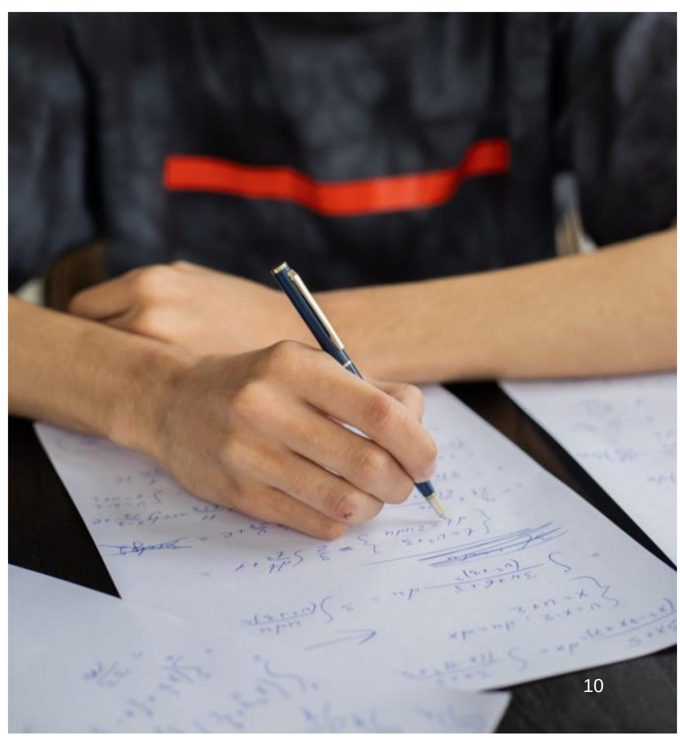
LAGRANGE's EQUATIONS



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Lagrange's equations

Hamilton's principle

$$\int_{t_1}^{t_2} (\delta L + \delta W_{nc})\, dt = 0 \qquad L = T - V$$

Generalized coordinates

$$\overline{x}_i = \overline{x}_i(q_1, ..., q_n; t) \longrightarrow \dot{\overline{x}}_i = \sum_k \frac{\partial \overline{x}_i}{\partial q_k}\dot{q}_k + \frac{\partial \overline{x}_i}{\partial t}$$

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i\, \dot{\overline{x}}_i \cdot \dot{\overline{x}}_i = T(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n; t) \qquad V = V(q_1, ..., q_n; t)$$

$$\delta W_{nc} = \sum_i \overline{F}_i \delta \overline{x}_i = \sum_i \sum_k \overline{F}_i \frac{\partial \overline{x}_i}{\partial q_k}\delta q_k = \sum_k Q_k \delta q_k$$

$$\longrightarrow \boxed{Q_k = \sum_i \overline{F}_i \frac{\partial \overline{x}_i}{\partial q_k}}$$

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Lagrange's equations

$$\int_{t_1}^{t_2} \left[\delta L + \sum_k Q_k \delta q_k\right] dt = 0$$

$$\int_{t_1}^{t_2} \left[\sum_k \left(\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\right) + \sum_k Q_k \delta q_k\right] dt = 0$$

$$\frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k$$

$$\longrightarrow \sum_k \left[\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_k \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} - Q_k\right] \delta q_k dt = 0$$

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Lagrange's equations

$$\sum_k \left[\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\sum_k\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} - Q_k\right]\delta q_k dt = 0$$

$$\overline{\delta q}_k(t_1) = \overline{\delta q}_k(t_2) = 0$$

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k \qquad k = 1, ...n$$

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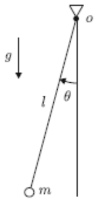
Lagrange's equations - examples



$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$$

$$\longrightarrow \boxed{ml^2\ddot{\theta} + mgl\sin\theta = 0}$$

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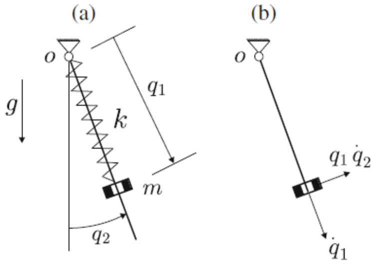
Lagrange's equations - examples

Pendulum with a sliding mass



$$T = \frac{1}{2}m\left(\dot{q_1}^2 + q_1^2\dot{q_2}^2\right)$$

$$V = -mgq_1\cos q_2 + \frac{1}{2}kq_1^2$$

$$L = T - V = \frac{1}{2}m\left(\dot{q_1}^2 + q_1^2\dot{q_2}^2\right) + mgq_1\cos q_2 - \frac{1}{2}kq_1^2$$

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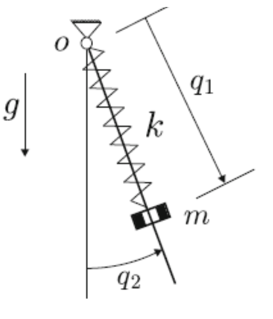
Lagrange's equations - examples

$$L = T - V = \frac{1}{2}m\left(\dot{q_1}^2 + q_1^2\dot{q_2}^2\right) + mgq_1\cos q_2 - \frac{1}{2}kq_1^2$$

$q_1 \longrightarrow m\ddot{q}_1 - mq_1\dot{q_2}^2 - mg\cos q_2 + kq_1 = 0$

$q_2 \longrightarrow \frac{d}{dt}\left(mq_1^2\dot{q}_2\right) + mgq_1\sin q_2 = 0$



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Lagrange's equations with constraints

Consider n generalized coordinates which are not independent

Constraint equations

$$\sum_k a_{lk}\delta q_k = 0 \qquad l = 1, ..., m$$

→ n-m degrees of freedom

$$\sum_k \left[\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\sum_k \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} - Q_k\right]\delta q_k dt = 0$$

δq_k Not independent → Lagrange multipliers

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$$

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Lagrange's equations with constraints

Lagrange multipliers

$$\sum_{l=1}^{m}\lambda_l\left(\sum_{k=1}^{n} a_{lk}\delta q_k\right) = \sum_{k=1}^{n}\delta q_k\left(\sum_{l=1}^{m}\lambda_l a_{lk}\right) = 0$$

$$\longrightarrow \int_{t_1}^{t_2}\sum_{k=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} - Q_k - \sum_{l=1}^{m}\lambda_l a_{lk}\right]\delta q_k dt = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k + \sum_{l=1}^{m}\lambda_l a_{lk}} \qquad k = 1, ..., n$$

Generalized contraint forces

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Lagrange's equations with constraints

n+m unknowns $q_k, \quad \lambda_l$

n+m equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k + \sum_{l=1}^{m} \lambda_l a_{lk} \qquad k = 1, ..., n$$

$$\sum_{k=1}^{n} a_{lk} \delta q_k = 0 \qquad l = 1, ..., m$$ constraint equations

Lagrange's equations with constraints

Holonomic system

$$f_l(q_1, ..., q_n, t) = 0 \longrightarrow \delta f_l = \sum_{k=1}^{n} \frac{\partial f_l}{\partial q_k} \delta q_k = 0 \longrightarrow a_{lk} = \frac{\partial f_l}{\partial q_k}$$

Non-holonomic system

$$\sum_{k=1}^{n} a_{lk} \delta q_k + a_{l0} dt = 0$$

LAGRANGE vs NEWTON

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Lagrange's equations with constraints

- Newton
  - 6 equations per rigid body
  - Constraints appear as forces

- Lagrange
  - n coordinates
  - m constraints / Lagrange multipliers
  - m+n equations of motion

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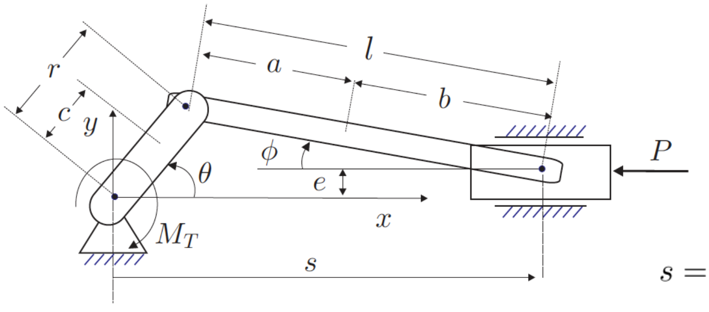
Piston engine



Coordinates

θ, ϕ, s n=3

Constraints

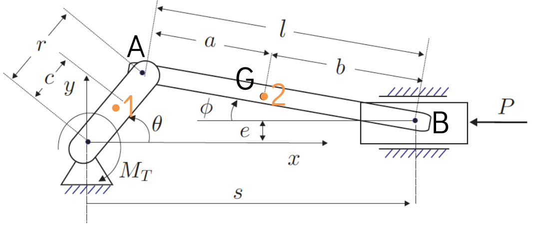
m=2

$$s = r\cos\theta + l\cos\phi$$
$$e = r\sin\theta - l\sin\phi$$

$$\lambda = \frac{r}{l} \longrightarrow \quad s = r\left(\cos\theta + \frac{1}{\lambda}\cos\phi\right)$$
$$\sin\phi = \left(\lambda\sin\theta - \frac{e}{l}\right)$$

Piston engine : kinetic energy

$$T_1 = \frac{1}{2}\left(J_1 + m_1c^2\right)\dot{\theta}^2 = \frac{1}{2}J_0\dot{\theta}^2$$
$$T_2 = \frac{1}{2}m_2v_G^2 + \frac{1}{2}J_2\dot{\phi}^2$$
$$T_3 = \frac{1}{2}m_3\dot{s}^2$$



Distribute mass at points A and B

$$m_A = \frac{m_2b}{l} \qquad m_B = \frac{m_2a}{l}$$

$$T_2 = \frac{1}{2}m_A\left(r\dot{\theta}\right)^2 + \frac{1}{2}m_B\dot{s}^2 + \frac{1}{2}J_{AB}\dot{\phi}^2 \qquad J_{AB} = J_2 - m_2ab$$

Piston engine : kinetic energy

Define $k_\phi = \frac{\dot{\phi}}{\dot{\theta}}$ $\qquad k_s = \frac{\dot{s}}{\dot{\theta}}$

$$\sin\phi = \left(\lambda\sin\theta - \frac{e}{l}\right) \rightarrow \cos\phi\dot{\phi} = \lambda\cos\theta\dot{\theta}$$

$$s = r\left(\cos\theta + \frac{1}{\lambda}\cos\phi\right)$$

$$\rightarrow \dot{s} = -r\left(\sin\theta\dot{\theta} + \frac{1}{\lambda}\sin\phi\dot{\phi}\right)$$

$$\rightarrow \frac{\dot{\phi}}{\dot{\theta}} = \frac{\lambda\cos\theta}{\cos\phi} = \frac{\lambda\cos\theta}{\sqrt{1-\left(\lambda\sin\theta - \frac{e}{l}\right)^2}}$$

$$\rightarrow \frac{\dot{s}}{\dot{\theta}} = -r\left(\sin\theta + \frac{k_\phi}{\lambda}\sin\phi\right)$$

$$T_1 = \frac{1}{2}\left(J_1 + m_1c^2\right)\dot{\theta}^2 = \frac{1}{2}J_0\dot{\theta}^2$$

$$\rightarrow T_2 = \frac{1}{2}m_A\left(r\dot{\theta}\right)^2 + \frac{1}{2}m_B\dot{s}^2 + \frac{1}{2}J_{AB}\dot{\phi}^2 = \left(\frac{1}{2}m_Ar^2 + \frac{1}{2}m_Bk_s^2 + \frac{1}{2}J_{AB}k_\phi^2\right)\dot{\theta}^2$$

$$T_3 = \frac{1}{2}m_3\dot{s}^2 = \frac{1}{2}m_3k_s^2\dot{\theta}^2$$

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Piston engine : kinetic energy

$$T = T_1 + T_2 + T_3 = \frac{1}{2}I(\theta)\dot{\theta}^2$$

$$I(\theta) = J_0 + m_Ar^2 + (m_3 + m_B)\,k_s^2 + J_{AB}k_\phi^2$$

Assume e=0, limit to first order terms

$$I(\theta) \simeq J_0 + m_Ar^2 + (m_3 + m_B)\,r^2\sin^2\theta + J_{AB}\lambda^2\cos^2\theta = A - B\cos 2\theta$$

$$A = J_0 + m_Ar^2 + \frac{1}{2}\left[(m_3 + m_B)\,r^2 + J_{AB}\lambda^2\right]$$

$$B = \frac{1}{2}\left[(m_3 + m_B)\,r^2 - J_{AB}\lambda^2\right]$$

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Piston engine : Lagrange equations

External forces:

$$\delta W_{nc} = Q_\theta \delta_\theta = -P\delta s - M_T \delta\theta = -\left(Pk_s + M_T\right)\delta\theta \qquad k_s = \frac{\dot{s}}{\dot{\theta}} = \frac{\delta s}{\delta\theta}$$

Lagrange equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_\theta \qquad L = T = \frac{1}{2} I(\theta)\dot{\theta}^2$

$$I(\theta)\ddot{\theta} + \frac{dI}{dt}\dot{\theta} - \frac{1}{2}\frac{dI}{d\theta}\dot{\theta}^2 = -Pk_s - M_T \qquad \frac{dI}{dt} = \frac{dI}{d\theta}\frac{d\theta}{dt}$$

$$\boxed{I(\theta)\ddot{\theta} + \frac{1}{2}\frac{dI}{d\theta}\dot{\theta}^2 = -Pk_s - M_T}$$

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Piston engine : Lagrange equations

Constant speed $\dot{\theta} = \omega_0$ and P= 0

$$I(\theta)\ddot{\theta} + \frac{1}{2}\frac{dI}{d\theta}\dot{\theta}^2 = -Pk_s - M_T \longrightarrow \frac{1}{2}\frac{dI}{d\theta}\omega_0^2 = -M_T$$

$$I(\theta) = A - B\cos 2\theta$$

$$\boxed{B\sin 2\theta\omega_0^2 = -M_T}$$

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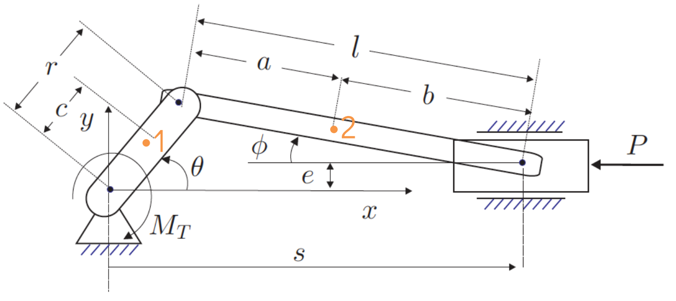
Piston engine : Reaction forces



Generalized coordinates:

$$x_1, y_1, x_2, y_2, \theta, \phi, s$$

$$\begin{aligned} s - r\cos\theta - l\cos\phi &= 0 = f_1 \\ e - r\sin\theta + l\sin\phi &= 0 = f_2 \\ x_1 - c\cos\theta &= 0 = f_3 \\ y_1 - c\sin\theta &= 0 = f_4 \\ x_2 - r\cos\theta - a\cos\phi &= 0 = f_5 \\ y_2 - b\sin\phi - e &= 0 = f_6 \end{aligned}$$

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Piston engine : Reaction forces

$$\begin{aligned} T = &\frac{1}{2} m_1 \left(\dot{x_1}^2 + \dot{y_1}^2\right) + \frac{1}{2} J_1 \dot{\theta}^2 \\ &+ \frac{1}{2} m_2 \left(\dot{x_2}^2 + \dot{y_2}^2\right) + \frac{1}{2} J_2 \dot{\phi}^2 \\ &+ \frac{1}{2} m_3 \dot{s}^2 \end{aligned}$$

$$\begin{aligned} s - r\cos\theta - l\cos\phi &= 0 = f_1 \\ e - r\sin\theta + l\sin\phi &= 0 = f_2 \\ x_1 - c\cos\theta &= 0 = f_3 \\ y_1 - c\sin\theta &= 0 = f_4 \\ x_2 - r\cos\theta - a\cos\phi &= 0 = f_5 \\ y_2 - b\sin\phi - e &= 0 = f_6 \end{aligned}$$

$$x_1(k=1): \quad \frac{d}{dt}(m_1 \dot{x_1}) = \sum_{l=1}^{6} a_{l1} \lambda_l$$

$$a_{l1} = \frac{\partial f_l}{\partial x_1}$$ → All = 0 except $a_{31} = 1$

→ $$\boxed{m_1 \ddot{x_1} = \lambda_3}$$

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Piston engine : Reaction forces

$$\rightarrow \quad \begin{aligned} m_1\ddot{x}_1 &= \lambda_3 \\ m_1\ddot{y}_1 &= \lambda_4 \\ m_2\ddot{x}_2 &= \lambda_5 \\ m_2\ddot{y}_2 &= \lambda_6 \end{aligned}$$

$$\begin{aligned} s - r\cos\theta - l\cos\phi &= 0 = f_1 \\ e - r\sin\theta + l\sin\phi &= 0 = f_2 \\ x_1 - c\cos\theta &= 0 = f_3 \\ y_1 - c\sin\theta &= 0 = f_4 \\ x_2 - r\cos\theta - a\cos\phi &= 0 = f_5 \\ y_2 - b\sin\phi - e &= 0 = f_6 \end{aligned}$$

$$J_1\ddot{\theta} = -M_T + (\lambda_1 + \lambda_5)\, r\sin\theta - \lambda_2 r\cos\theta + c\,(\lambda_3\sin\theta - \lambda_4\cos\theta)$$

$$J_2\ddot{\phi} = l\,(\lambda_1\sin\phi + \lambda_2\cos\phi) + \lambda_5 a\sin\phi - \lambda_6 b\cos\phi$$

$$m_3\ddot{s}_3 = \lambda_1 - P$$

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Piston engine : Reaction forces

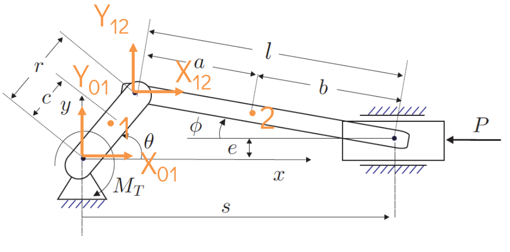
Newton's second law :

$$m_1\ddot{x}_1 = X_{01} + X_{12} \longrightarrow \lambda_3 = X_{01} + X_{12}$$

$$m_1\ddot{y}_1 = Y_{01} + Y_{12} \longrightarrow \lambda_4 = Y_{01} + Y_{12}$$

$$J_1\ddot{\theta} = -M_T + X_{01}c\sin\theta - Y_{01}c\cos\theta - X_{12}(r-c)\sin\theta + Y_{12}(r-c)\cos\theta$$

$$J_1\ddot{\theta} = -M_T + (\lambda_1 + \lambda_5)\, r\sin\theta - \lambda_2 r\cos\theta + c\,(\lambda_3\sin\theta - \lambda_4\cos\theta)$$



$$\longrightarrow \quad \begin{aligned} -X_{12} &= \lambda_1 + \lambda_5 \\ Y_{12} &= -\lambda_2 \end{aligned}$$

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