Multiple DOF systems

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MDOF SYSTEMS IN REAL LIFE

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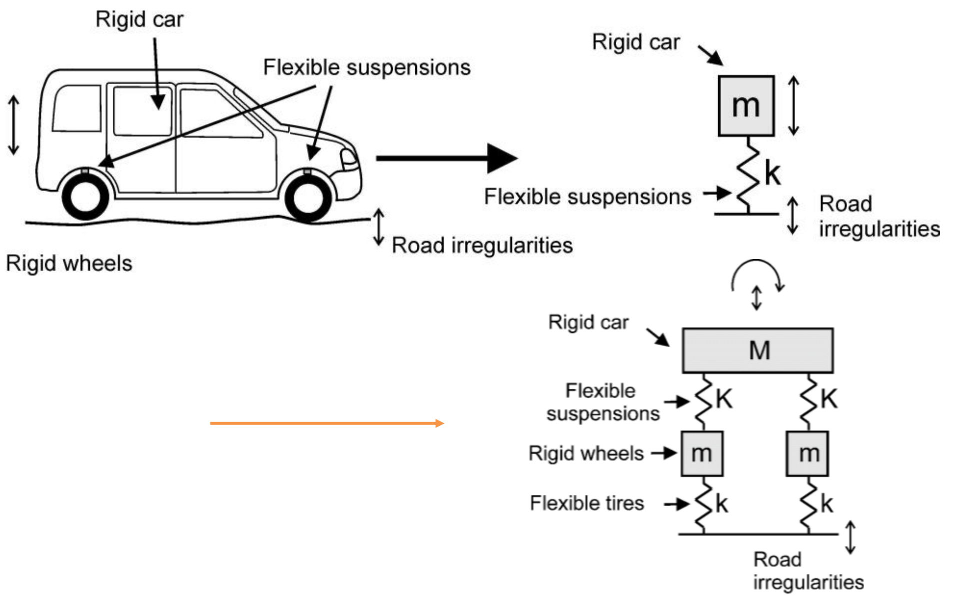
From SDOF to MDOF



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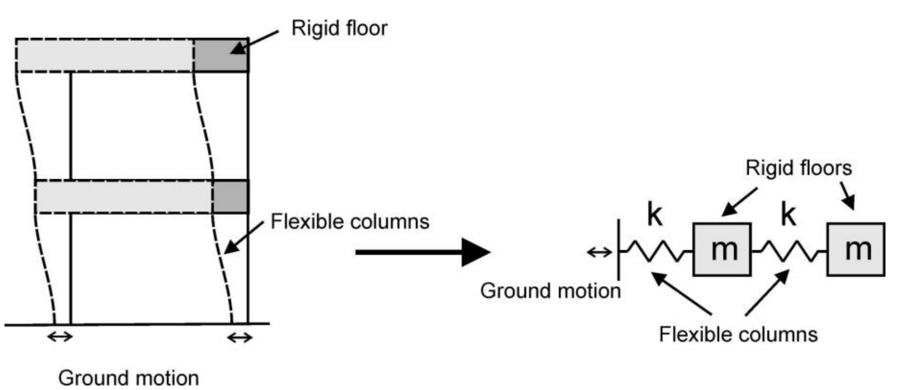
MDOF systems in real life



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UNDAMPED RESPONSE

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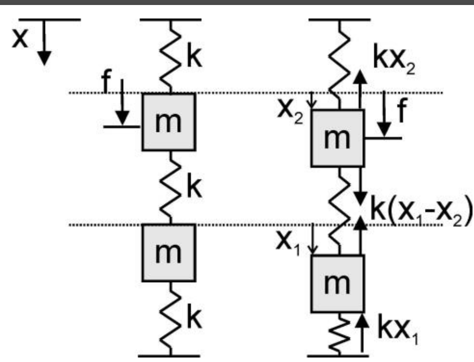
Equations of motion



$$
\begin{aligned}
m\ddot{x}_1 &= -kx_1 - k\left(x_1 - x_2\right) \\
m\ddot{x}_2 &= k\left(x_1 - x_2\right) - kx_2 + f
\end{aligned}
$$

$$
\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}
$$

$$
M\ddot{x} + Kx = F
$$

Mass matrix

Stiffness matrix

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Free response

$$M\ddot{x} + Kx = 0$$

$$\left\{ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right\} = \left\{ \begin{array}{c} A_1 \\ A_2 \end{array} \right\} e^{rt} = \psi e^{rt}$$

$$\left(K + r^2 M\right) \psi = 0$$

Admits a non trivial solution if

$$det(K + r^2 M) = 0$$

r^2 is negative (K and M are positive definite matrices)

$$r^2 = -\omega^2$$

$$\left(K - \omega^2 M\right) \psi = 0$$

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Free response

$$\left(K - \omega^2 M\right) \psi = 0$$

Generalized eigenvalue problem ($-\omega^2$)

$$det(K - \omega^2 M) = 0$$

If the system has *n* degrees of freedom, there exist *n* values of $-\omega^2$ for which this equation is satisfied. These are the *n* eigenvalues which correspond to *n* eigenfrequencies

n eigen vectors ψ are associated to these eigenfrequencies. They correspond to the *n* mode shapes of the structure

The general solution is written in the form:

$$x(t) = \sum_{i=1}^{n} \left(Z_{i1} cos(\omega_i t) + Z_{i2} sin(\omega_i t)\right) \psi_i$$

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Mode shapes orthogonality

Property :

$$\begin{aligned} \psi_i^T M \psi_j &= \delta_{ij}\mu_i \\ \psi_i^T K \psi_j &= \delta_{ij}\mu_i\omega_i^2 \end{aligned}$$

Proof :

$$\left(K - \omega_i^2 M\right)\psi_i = 0 \qquad (1)$$

$$\left(K - \omega_j^2 M\right)\psi_j = 0 \qquad (2)$$

$$\omega_i \neq \omega_j$$

Premultiply (1) by ψ_j^T ,(2) by ψ_i^T and substract taking into account symmetry of K ($\psi_i^T K\psi_j = \psi_j^T K\psi_i$) and M ($\psi_i^T M\psi_j = \psi_j^T M\psi_i$)

$$\longrightarrow \psi_j^T M\psi_i(\omega_j^2 - \omega_i^2) = 0 \qquad i \neq j$$

$$\longrightarrow \psi_j^T M\psi_i = 0 \qquad i \neq j$$

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Mode shapes orthogonality

$$\psi_j^T M\psi_i = 0 \qquad i \neq j$$

Define $\mu_i = \psi_i^T M\psi_i$ $\longrightarrow$ $\psi_i^T M\psi_j = \delta_{ij}\mu_i$

$K\psi_i = \omega_i^2 M\psi_i$ $\longrightarrow$ $\psi_i^T K\psi_j = \delta_{ij}\mu_i\omega_i^2$

Matrix notation

$$\Psi = \begin{bmatrix} \psi_1 & \psi_2 & \dots & \psi_n \end{bmatrix}$$

$$\longrightarrow \begin{aligned} \Psi^T M\Psi &= diag(\mu_i) \\ \Psi^T K\Psi &= diag(\mu_i\omega_i^2) \end{aligned}$$

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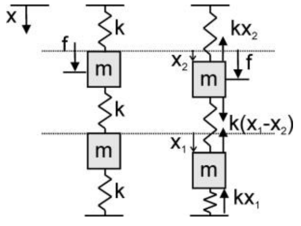
Example of a 2 DOFs system



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

$$\boxed{\left(K - \omega^2 M\right) \psi = 0}$$

$$det(K - \omega^2 M) = det \begin{pmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{pmatrix} = 0$$

$$(2k - \omega^2 m)(2k - \omega^2 m) - k^2 = m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$$

Second order equation in ω^2 →

$$\begin{aligned} \omega_1^2 &= k/m \\ \omega_2^2 &= 3k/m \end{aligned}$$

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Example of a 2 DOFs system

$$\boxed{\left(K - \omega^2 M\right) \psi = 0}$$

$$(K - \omega^2 M) = \begin{bmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{bmatrix}$$

$$\psi = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$$

For $\omega_1^2 = k/m$

$$\left(2k - \frac{k}{m} m\right) A_1 - kA_2 = 0$$

$$kA_1 = kA_2 \Rightarrow A_1 = A_2$$

For $\omega_2^2 = 3k/m$

$$\left(2k - \frac{3k}{m} m\right) A_1 - kA_2 = 0$$

$$-kA_1 = kA_2 \Rightarrow A_1 = -A_2$$

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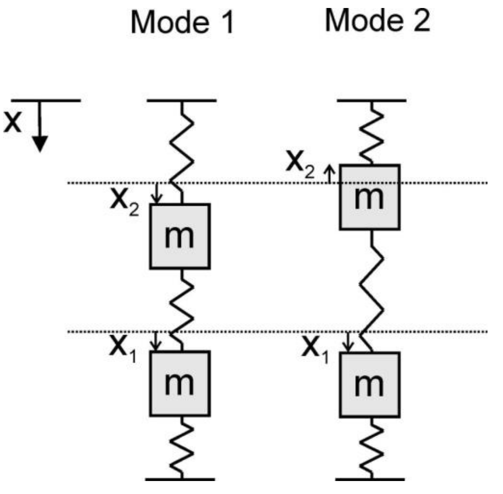
Example of a 2 DOFs system

$$\omega_1^2 = k/m \qquad \psi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\omega_2^2 = 3k/m \qquad \psi_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$



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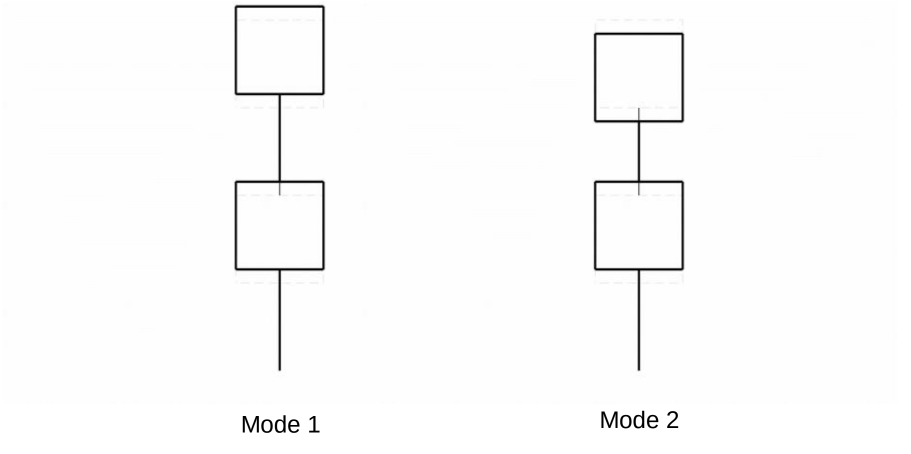
Example of a 2 DOFs system



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Example of a 2 DOFs system

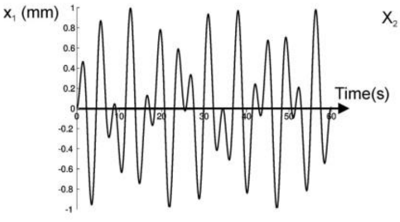
$$\left\{ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right\} = (Z_{11}\cos\omega_1 t + Z_{12}\sin\omega_1 t)\left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} + (Z_{21}\cos\omega_2 t + Z_{22}\sin\omega_2 t)\left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\}$$

Assume the following initial conditions

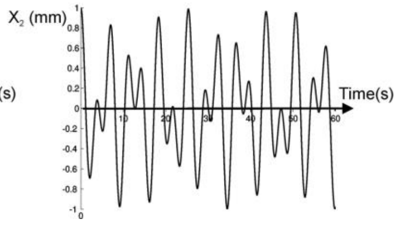
$$\left\{ \begin{array}{c} x_1(0) \\ x_2(0) \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 1mm \end{array} \right\} \qquad \dot{x}_1(0) = \dot{x}_2(0) = 0$$

$$\longrightarrow \left\{ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right\} = \left(\begin{array}{c} \frac{1}{2}\cos\omega_1 t - \frac{1}{2}\cos\omega_2 t \\ \frac{1}{2}cos\omega_1 t + \frac{1}{2}cos\omega_2 t \end{array} \right) (mm)$$

Mass 1



Mass 2



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Resonance of MDOF systems

https://youtu.be/OaXSmPgl1os

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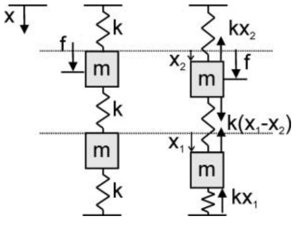
Harmonic excitation



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

$$M\ddot{x} + Kx = f \qquad \begin{aligned} x(t) &= Xe^{i\omega t} \\ f(t) &= Fe^{i\omega t} \end{aligned}$$

$$\downarrow$$

$$\boxed{\left(K - \omega^2 M\right) X = F}$$

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Example of a 2 DOFs system

Response to harmonic excitation $\quad x_1(t) = X_1 e^{i\omega t} \quad x_2(t) = X_2 e^{i\omega t} \qquad f(t) = Fe^{i\omega t}$

$$\begin{bmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

$$X_1/F = \frac{k}{\boxed{(2k - \omega^2 m)^2 - k^2}} \longrightarrow \begin{aligned} \omega_1^2 &= k/m \\ \omega_2^2 &= 3k/m \end{aligned}$$

Resonance

$$X_2/F = \frac{\boxed{2k - \omega^2 m}}{(2k - \omega^2 m)^2 - k^2} \longrightarrow \omega_0^2 = 2k/m$$

Anti-resonance

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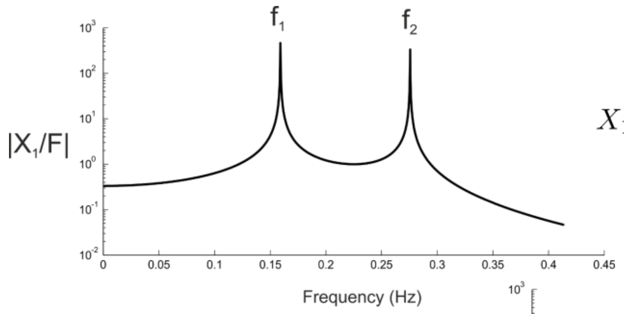
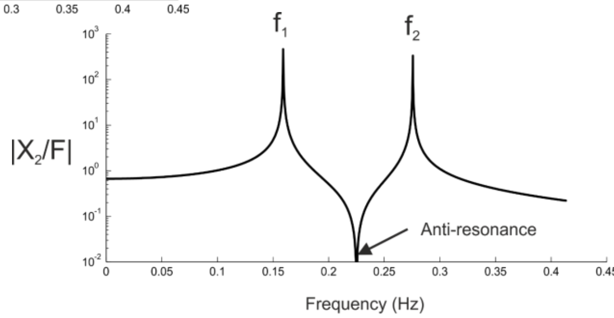
Example of a 2 DOFs system



$$X_1/F = \frac{-k}{(2k-\omega^2 m)^2 - k^2}$$

(m=1kg, k = 1N/m)

$$X_2/F = \frac{2k-\omega^2 m}{(2k-\omega^2 m)^2 - k^2}$$



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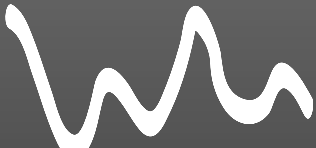
DAMPED RESPONSE



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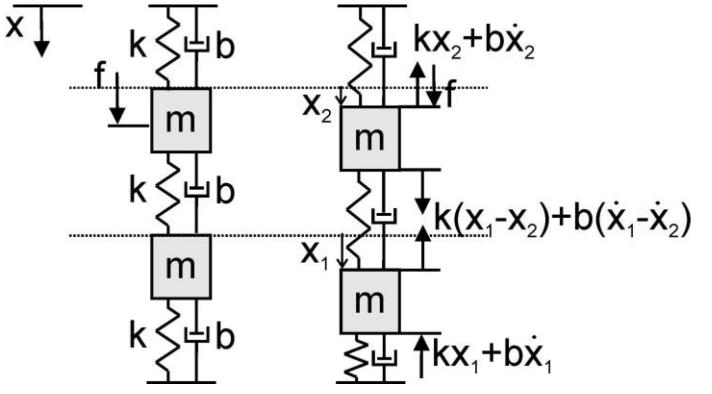
Equations of motion



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2b & -b \\ -b & 2b \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

$$M\ddot{x} + C\dot{x} + Kx = f$$

Damping matrix

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Free response

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} e^{rt} = \psi e^{rt}$$

$$\left(K + rC + r^2 M\right)\psi = 0$$

Non trivial solution if

$$det(K + rC + r^2 M) = 0$$

- Complex roots of the characteristic equation
  -> Oscillatory functions with exponential envelope
- Complex eigen vectors = complex modeshapes
  -> Not often used in practice in vibrations

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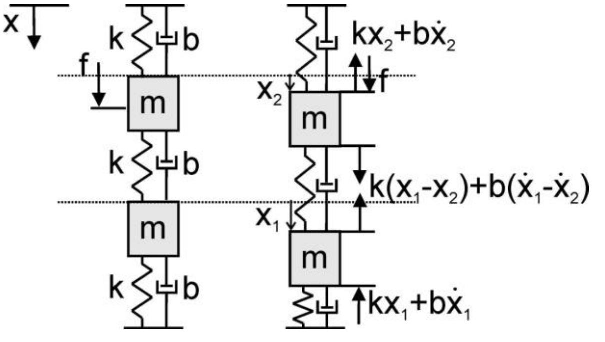
Harmonic excitation



$$M\ddot{x} + C\dot{x} + Kx = f \qquad x(t) = Xe^{i\omega t} \qquad f(t) = Fe^{i\omega t}$$

$$\left(K + i\omega C - \omega^2 M\right) X = F$$

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Example of a 2 DOFs system

Response to harmonic excitation $x_1(t) = X_1 e^{i\omega t}$ $f(t) = Fe^{i\omega t}$
$x_2(t) = X_2 e^{i\omega t}$

$$\begin{bmatrix} 2k + 2i\omega b - \omega^2 m & -(k + i\omega b) \\ -(k + i\omega b) & 2k + 2i\omega b - \omega^2 m \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

$$X_1/F = \frac{k + i\omega b}{\boxed{(2k + 2i\omega b - \omega^2 m)^2 - (k + i\omega b)^2}}$$

→ Damped resonances

$$X_2/F = \frac{\boxed{2k + 2i\omega b - \omega^2 m}}{(2k + 2i\omega b - \omega^2 m)^2 - (k + i\omega b)^2}$$

→ No strict anti-resonance

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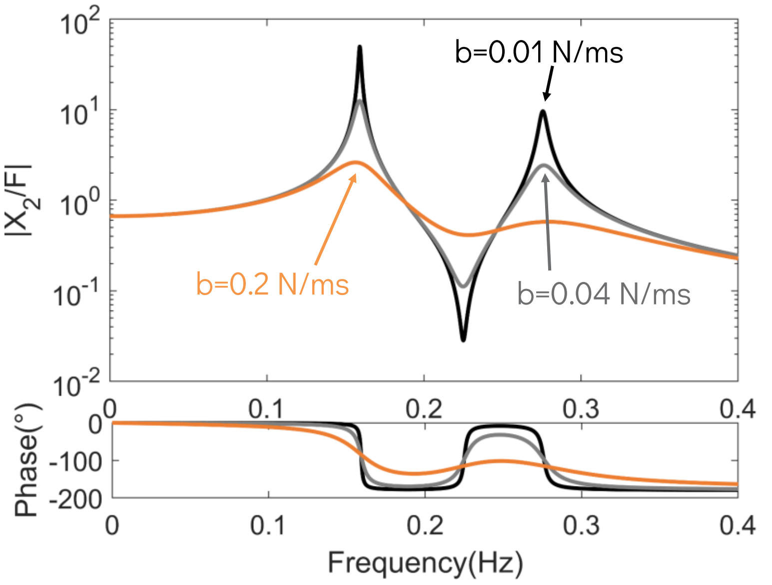
Example of a 2 DOFs system



(m=1kg, k = 1N/m)

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BASE EXCITATION

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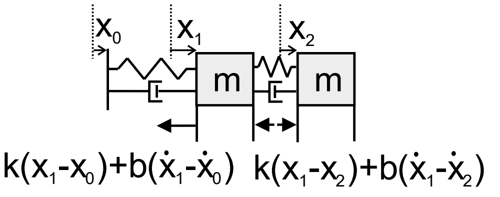
Base excitation of MDOF systems



Equations of motion:

$$
\begin{aligned}
m\ddot{x}_1 &= -k(x_1 - x_0) - b(\dot{x}_1 - \dot{x}_0) - k(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) \\
m\ddot{x}_2 &= k(x_1 - x_2) + b(\dot{x}_1 - \dot{x}_2)
\end{aligned}
$$

$$
\begin{aligned}
x_{1r} &= x_1 - x_0 \\
x_{2r} &= x_2 - x_0
\end{aligned}
\longrightarrow
\begin{aligned}
m\ddot{x}_{1r} + 2b\dot{x}_{1r} - b\dot{x}_{2r} + 2kx_{1r} - kx_{2r} &= -m\ddot{x}_0 \\
m\ddot{x}_{2r} + b\dot{x}_{2r} - b\dot{x}_{1r} + kx_{2r} - kx_{1r} &= -m\ddot{x}_0
\end{aligned}
$$

$$
\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_{1r} \\ \ddot{x}_{2r} \end{Bmatrix} + \begin{bmatrix} 2b & -b \\ -b & b \end{bmatrix} \begin{Bmatrix} \dot{x}_{1r} \\ \dot{x}_{2r} \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_{1r} \\ x_{2r} \end{Bmatrix} = \begin{Bmatrix} -m\ddot{x}_0 \\ -m\ddot{x}_0 \end{Bmatrix}
$$

Base excitation of MDOF systems

$$
\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_{1r} \\ \ddot{x}_{2r} \end{Bmatrix} + \begin{bmatrix} 2b & -b \\ -b & b \end{bmatrix} \begin{Bmatrix} \dot{x}_{1r} \\ \dot{x}_{2r} \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_{1r} \\ x_{2r} \end{Bmatrix} = \begin{Bmatrix} -m\ddot{x}_0 \\ -m\ddot{x}_0 \end{Bmatrix}
$$

Matrix notations:

$$M\ddot{x}_r + C\dot{x}_r + Kx_r = -M\ddot{x}_b$$

$$x_r = \begin{Bmatrix} x_1 - x_0 \\ x_2 - x_0 \end{Bmatrix} \qquad \ddot{x}_b = \begin{Bmatrix} \ddot{x}_0 \\ \ddot{x}_0 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ddot{x}_0 = T\,\ddot{x}_0$$

→ All developments for force excitation apply