Rotor dynamics

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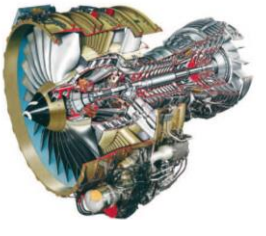
Machines with rotors

Refrigeration compressor rotor

Jet engines



Machine tools

Electricity power plant

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JEFCOTT ROTOR WITHOUT DAMPING

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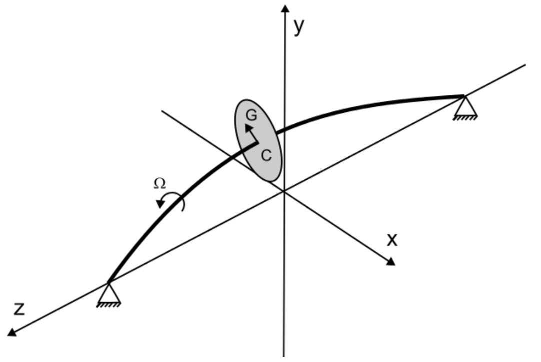
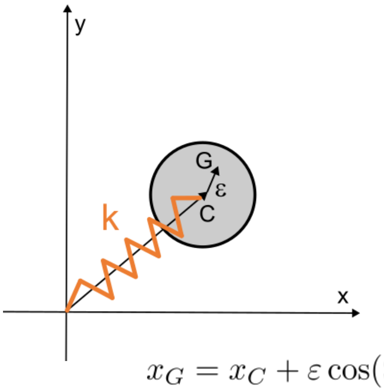
Equivalent model





$$x_G = x_C + \varepsilon \cos(\Omega t)$$
$$y_G = y_C + \varepsilon \sin(\Omega t)$$

Equivalent stiffness of the shaft ?

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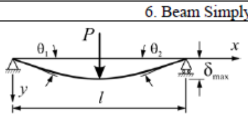
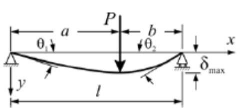
Equivalent stiffness of beams

BEAM TYPE	SLOPE AT ENDS	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM AND CENTER DEFLECTION
6. Beam Simply Supported at Ends – Concentrated load P at the center			
	$\theta_1 = \theta_2 = \frac{Pl^2}{16EI}$	$y = \frac{Px}{12EI}\left(\frac{3l^2}{4} - x^2\right)$ for $0 < x < \frac{l}{2}$	$\delta_{max} = \frac{Pl^3}{48EI}$
7. Beam Simply Supported at Ends – Concentrated load P at any point			
	$\theta_1 = \frac{Pb(l^2 - b^2)}{6lEI}$ $\theta_2 = \frac{Pab(2l - b)}{6lEI}$	$y = \frac{Pbx}{6lEI}\left(l^2 - x^2 - b^2\right)$ for $0 < x < a$ $y = \frac{Pb}{6lEI}\left[\frac{l}{b}(x-a)^3 + \left(l^2 - b^2\right)x - x^3\right]$ for $a < x < l$	$\delta_{max} = \frac{Pb\left(l^2 - b^2\right)^{3/2}}{9\sqrt{3}\,lEI}$ at $x = \sqrt{\left(l^2 - b^2\right)/3}$ $\delta = \frac{Pb}{48EI}\left(3l^2 - 4b^2\right)$ at the center, if $a > b$

http://home.eng.iastate.edu/~shermanp/STAT447/STAT%20Articles/Beam_Deflection_Formulae.pdf

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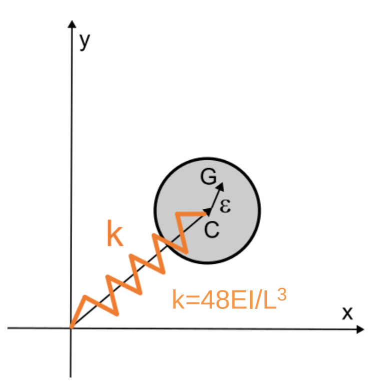
Equivalent model

$$x_G = x_C + \varepsilon\cos(\Omega t) \qquad \ddot{x_G} = \ddot{x_C} - \varepsilon\Omega^2\cos(\Omega t)$$
$$y_G = y_C + \varepsilon\sin(\Omega t) \qquad \ddot{y_G} = \ddot{y_C} - \varepsilon\Omega^2\sin(\Omega t)$$

$$|F_k| = k\sqrt{x_C^2 + y_C^2}$$

$$F_{kx} = -k\sqrt{x_C^2 + y_C^2}\,\frac{x_C}{\sqrt{x_C^2 + y_C^2}} = -kx_C$$
$$F_{ky} = -k\sqrt{x_C^2 + y_C^2}\,\frac{y_C}{\sqrt{x_C^2 + y_C^2}} = -ky_C$$

$$m\ddot{x}_C + kx_C = m\varepsilon\Omega^2\cos(\Omega t)$$
$$m\ddot{y}_C + ky_C = m\varepsilon\Omega^2\sin(\Omega t)$$



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Free whirling

$$m\ddot{x}_C + kx_C = 0$$
$$x_C = X_0 e^{st}$$

$$m\ddot{y}_C + ky_C = 0$$
$$y_C = Y_0 e^{st}$$

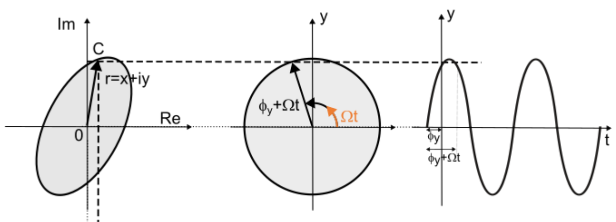
$$(ms^2 + k)X_0 e^{st} = 0 \qquad \omega_0 = \sqrt{k/m} \qquad (ms^2 + k)Y_0 e^{st} = 0$$

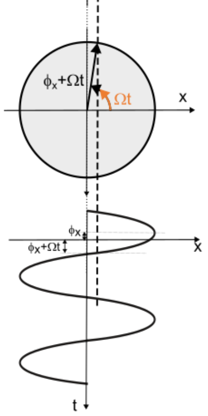
$$x_C(t) = X_1 e^{i\omega_0 t} + X_2 e^{-i\omega_0 t}$$
$$x_C(t) = x_C(0)\cos(\omega_0 t) + \frac{1}{\omega_0}\dot{x}_C(0)\sin(\omega_0 t)$$

$$y_C(t) = Y_1 e^{i\omega_0 t} + Y_2 e^{-i\omega_0 t}$$
$$y_C(t) = y_C(0)\cos(\omega_0 t) + \frac{1}{\omega_0}\dot{y}_C(0)\sin(\omega_0 t)$$

Equations fully decoupled

Complex coordinates





$$r_C(t) = x_C(t) + iy_C(t)$$

$$m\ddot{x}_C + kx_C = m\varepsilon\Omega^2\cos(\Omega t) \qquad (1)$$
$$m\ddot{y}_C + ky_C = m\varepsilon\Omega^2\sin(\Omega t) \qquad (2)$$

→ $(1) + i(2)$

→ $$m\ddot{r}_C + kr_C = m\varepsilon\Omega^2 e^{i\Omega t}$$

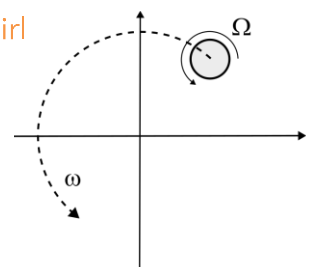
Free whirling

$$m\ddot{r}_C + kr_C = m\varepsilon\Omega^2 e^{i\Omega t}$$

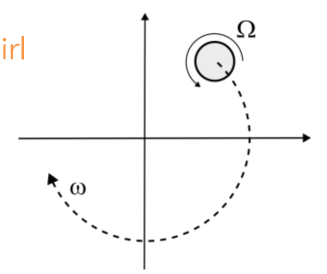
$$r_C(t) = R_0 e^{st} \longrightarrow (ms^2 + k)R_0 e^{st} = 0 \qquad s = \pm i\sqrt{\frac{k}{m}}$$

$$r_C(t) = R_1 e^{i\omega_0 t} + R_2 e^{-i\omega_0 t}$$

Forward whirl



Backward whirl



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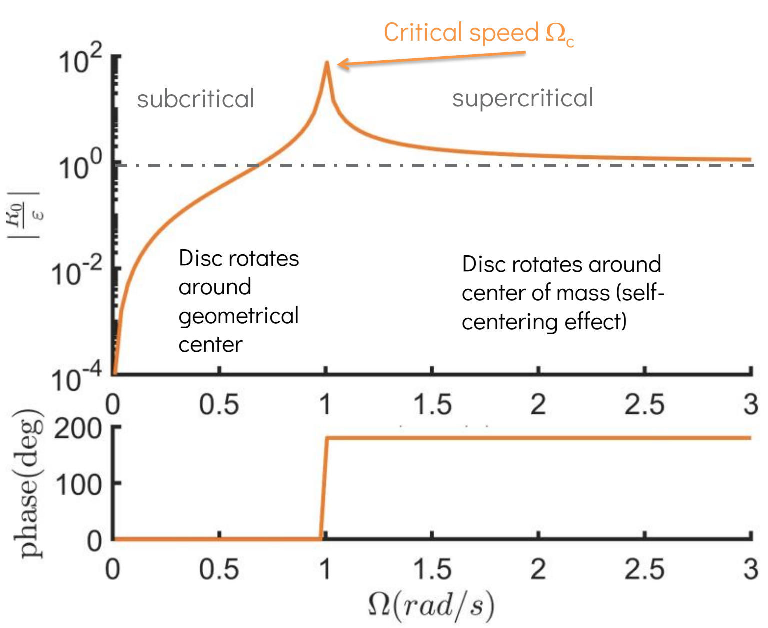
Unbalance (harmonic) response



$$r_C(t) = R_0 e^{i\Omega t}$$

$$(-m\Omega^2 + k)R_0 = m\varepsilon\Omega^2$$

$$R_0 = \varepsilon\frac{\Omega^2}{\Omega_c^2 - \Omega^2}$$

$$\Omega_c = \sqrt{\frac{k}{m}}$$

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Asymptotic high rotation speed response

$$R_0 = \varepsilon \frac{\Omega^2}{\Omega_c^2 - \Omega^2}$$

High rotating speed

$$\Omega >> \Omega_c \longrightarrow R_0 = -\varepsilon$$

$$r_C(t) = R_0 e^{i\Omega t} \longrightarrow r_C(t) = -\varepsilon\,(\cos\Omega t + i\sin\Omega t)$$

$$r_C(t) = x_C(t) + i\,y_C(t)$$

$$x_C(t) = -\varepsilon\cos(\Omega t)$$
$$y_C(t) = -\varepsilon\sin(\Omega t)$$

$$x_G = x_C + \varepsilon\cos(\Omega t)$$
$$y_G = y_C + \varepsilon\sin(\Omega t) \longrightarrow (x_G, y_G) = (0,0)$$

Disk rotates around center of gravity

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Estimation of the critical speed

- When both the mass and static stiffness are known : $\Omega_c = \sqrt{k/m}$
- Measure the free response to a hammer test
- Measure the static deflection $k = mg/y_0$
- Rotate the shaft and increase speed (dangerous)

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Estimation of the critical speed

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JEFCOTT ROTOR WITH VISCOUS DAMPING

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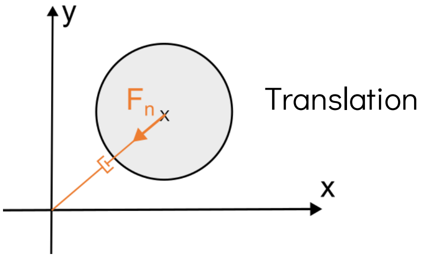
Damping models

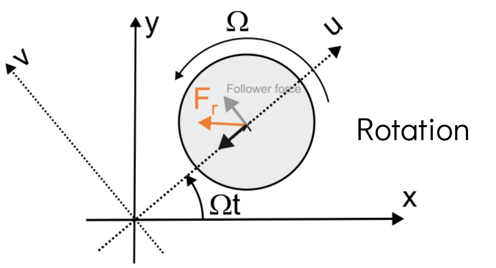
Non-rotating damping

$$F_n = \left\{ \begin{array}{c} F_x \\ F_y \end{array} \right\} = -c_n \left\{ \begin{array}{c} \dot{x}_C \\ \dot{y}_C \end{array} \right\}$$



Rotating damping

$$F_r = \left\{ \begin{array}{c} F_u \\ F_v \end{array} \right\} = -c_r \left\{ \begin{array}{c} \dot{u}_C \\ \dot{v}_C \end{array} \right\}$$



Jefcott rotor with damping

Express rotation damping in fixed frame :

$$\left\{ \begin{array}{c} u_c \\ v_c \end{array} \right\} = \mathbf{R} \left\{ \begin{array}{c} x_C \\ y_C \end{array} \right\} \qquad \mathbf{R} = \left[\begin{array}{cc} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{array} \right]$$

$$\left\{ \begin{array}{c} \dot{u}_c \\ \dot{v}_c \end{array} \right\} = \mathbf{R} \left\{ \begin{array}{c} \dot{x}_C \\ \dot{y}_C \end{array} \right\} + \dot{\mathbf{R}} \left\{ \begin{array}{c} x_C \\ y_C \end{array} \right\} \qquad \dot{\mathbf{R}} = \mathbf{\Omega} \left[\begin{array}{cc} -\sin(\Omega t) & \cos(\Omega t) \\ -\cos(\Omega t) & -\sin(\Omega t) \end{array} \right]$$

Follower force

$$\begin{aligned} \dot{u}_C &= \cos(\Omega t)\dot{x}_C + \sin(\Omega t)\dot{y}_C - \Omega \sin(\Omega t) x_C + \Omega \cos(\Omega t) y_C \\ \dot{v}_C &= -\sin(\Omega t)\dot{x}_C + \cos(\Omega t)\dot{y}_C - \Omega \cos(\Omega t) - \Omega \sin(\Omega t) y_C \end{aligned}$$

Damping models

$$F_r = \left\{ \begin{array}{c} F_u \\ F_v \end{array} \right\} = -c_r \left\{ \begin{array}{c} \dot{u}_C \\ \dot{v}_C \end{array} \right\}$$

$$F_u = -c_r \left(\cos(\Omega t)\dot{x}_C + \sin(\Omega t)\dot{y}_C - \Omega \sin(\Omega t) x_C + \Omega \cos(\Omega t) y_C \right)$$
$$F_v = -c_r \left(-\sin(\Omega t)\dot{x}_C + \cos(\Omega t)\dot{y}_C - \Omega \cos(\Omega t) x_C - \Omega \sin(\Omega t) y_C \right)$$

$$F_x = F_u \cos(\Omega t) - F_v \sin(\Omega t)$$
$$F_y = F_u \sin(\Omega t) + F_v \cos(\Omega t)$$

$$F_x = -c_r \left(\dot{x}_C + \Omega y_C \right)$$
$$F_y = -c_r \left(\dot{y}_C - \Omega x_C \right)$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \left\{ \begin{array}{c} \ddot{x}_c \\ \ddot{y}_c \end{array} \right\} + \begin{bmatrix} c_n + c_r & 0 \\ 0 & c_n + c_r \end{bmatrix} \left\{ \begin{array}{c} \dot{x}_c \\ \dot{y}_c \end{array} \right\} + \left(\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} + \Omega \begin{bmatrix} 0 & c_r \\ -c_r & 0 \end{bmatrix} \right) \left\{ \begin{array}{c} x_c \\ y_c \end{array} \right\} = m\varepsilon\Omega^2 \left\{ \begin{array}{c} \cos(\Omega t) \\ \sin(\Omega t) \end{array} \right\}$$

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Jefcott rotor with damping

(First line) + i (Second line)

$$r_C = x_C + i y_C$$

→ $$m\ddot{r}_C + (c_r + c_n)\dot{r}_C + (k - i c_r \Omega) r_C = m\varepsilon\Omega^2 e^{i\Omega t}$$

→ Complex stiffness
= negative damping

Rotating damping has tow effects:

1. dissipating energy
2. transferring energy from the rotation of the system to its vibration

→ Possibility of instability

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Jefcott rotor with damping : free whirling

$$r_C = r_0 e^{st} \longrightarrow ms^2 + (c_r + c_n)s + k - i\Omega c_r = 0$$

$$s = \sigma + i\omega = -\frac{c_r + c_n}{2m} \pm \sqrt{\frac{(c_r + c_n)^2 - 4m(k - i\Omega c_r)}{4m^2}}$$

$$r_C = R_1 e^{(\sigma_1 + i\omega_1)t} + R_2 e^{(\sigma_2 + i\omega_2)t}$$

$$\sqrt{a + ib} = \pm\left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\frac{|b|}{b}\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right)$$

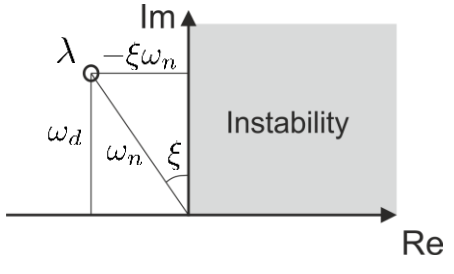
$$a = \frac{(c_r + c_n)^2 - 4mk}{4m^2} = -\Gamma \qquad b = \frac{\Omega c_r}{m}$$

$$\sigma_{1,2} = -\frac{c_r + c_n}{2m} \pm \frac{1}{\sqrt{2}}\sqrt{\sqrt{\Gamma^2 + \left(\frac{\Omega c_r}{m}\right)^2} - \Gamma}$$

$$\omega_{1,2} = \pm sgn(\Omega)\frac{1}{\sqrt{2}}\sqrt{\sqrt{\Gamma^2 + \left(\frac{\Omega c_r}{m}\right)^2} + \Gamma}$$

Complex poles and stability

Complex pole of the system



One DOF system impulse response

$$h(t) = \frac{e^{-\xi\omega_n t}}{m\omega_d} sin(\omega_d t)$$

Pole-residue model in the time domain is :

$$h(t) = Re^{\lambda t} + R^* e^{\lambda^* t}$$

$$\lambda = -\xi\omega_n + j\omega_d$$

Instability occurs when pole crosses the imaginary axis (negative damping)

Jefcott rotor with damping : stability

$$\omega_{1,2} = \pm sgn(\Omega)\frac{1}{\sqrt{2}}\sqrt{\sqrt{\Gamma^2 + \left(\frac{\Omega c_r}{m}\right)^2} + \Gamma}$$

Damped frequency depends on rotation speed

$$\sigma_{1,2} = -\frac{c_r + c_n}{2m} \pm \frac{1}{\sqrt{2}}\sqrt{\sqrt{\Gamma^2 + \left(\frac{\Omega c_r}{m}\right)^2} - \Gamma}$$

Decay rate depends on rotation speed

The system is stable (σ_1<0) if $\Omega < \sqrt{\frac{k}{m}}(1 + \frac{c_n}{c_r})$

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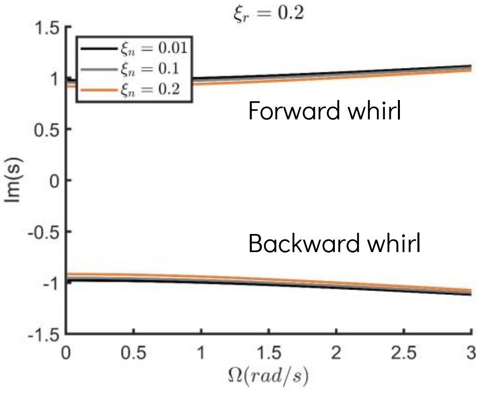
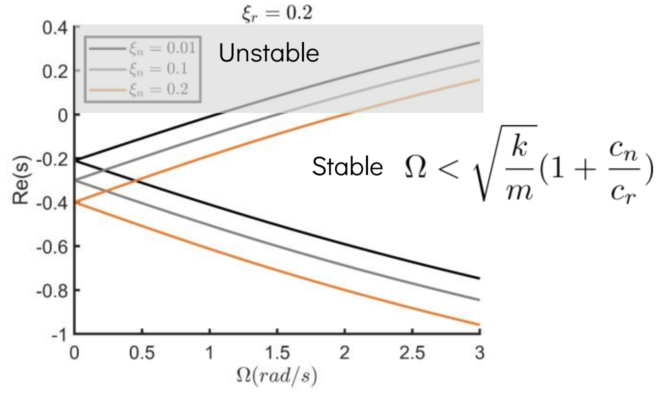
Jefcott rotor with damping : free whirling

Campbell diagram



Decay rate plot



For increasing rotation speed, the forward whirl becomes unstable after a certain rotation speed !

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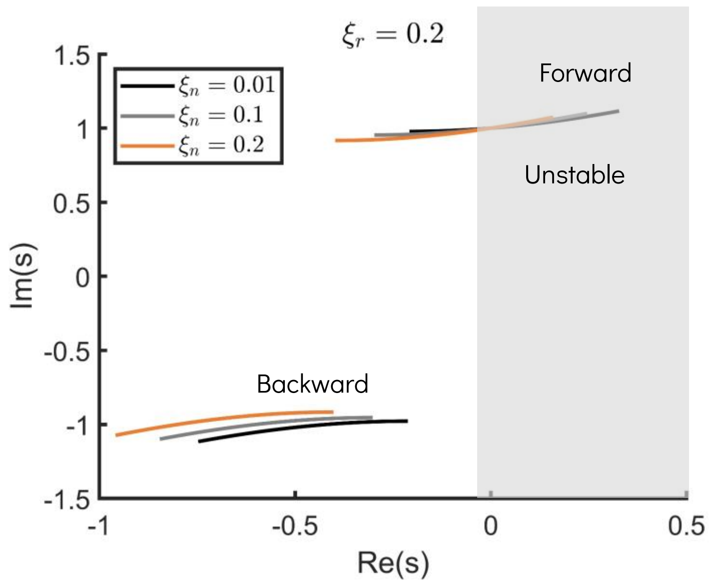
Jefcott rotor with damping : root locus

Evolution of the poles in the complex plane with increasing rotor speed

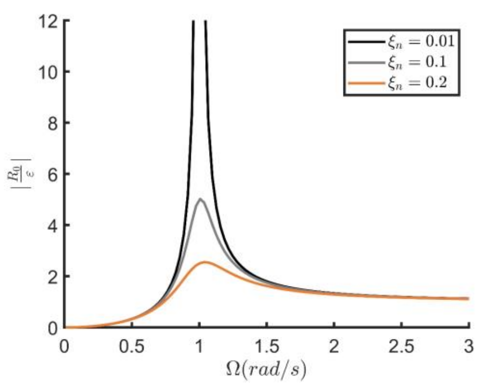


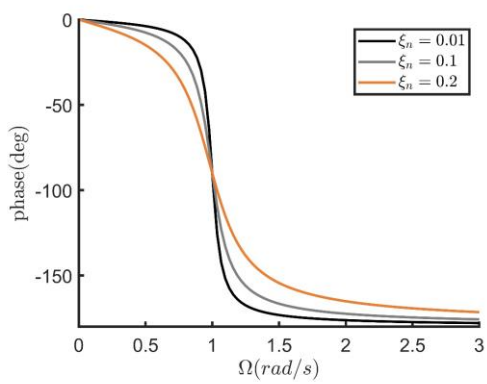
Jefcott rotor with damping : unbalance response

$$m\ddot{r}_C + (c_r + c_n)\dot{r}_C + (k - ic_r\Omega)r_C = m\varepsilon\Omega^2 e^{i\Omega t} \qquad r_C = R_0 e^{i\Omega t}$$

$$\boxed{R_0(-m\Omega^2 + i\Omega c_n + k) = m\varepsilon\Omega^2}$$

Only translation damping





Interpretation of harmonic response

$$r_C(t) = R_0 e^{i\omega t} = (R_r + iR_i)(\cos(\Omega t) + i\sin(\Omega t))$$

$$= (R_r\cos(\Omega t) - R_i\sin(\Omega t)) + i\,(R_i\cos(\Omega t) + R_r\sin(\Omega t))$$

$$x_C(t) = R_r\cos(\Omega t) - R_i\sin(\Omega t) \qquad y_C(t) = R_i\cos(\Omega t) + R_r\sin(\Omega t)$$

$$x_C(t) = R_r\cos(\Omega t + \phi) \qquad \phi = \tan^{-1}\left(\frac{R_i}{R_r}\right)$$

$$y_C(t) = R_r\sin(\Omega t + \phi) \qquad r_{circ} = \sqrt{R_r^2 + R_i^2}$$

Trajectory is always a circle

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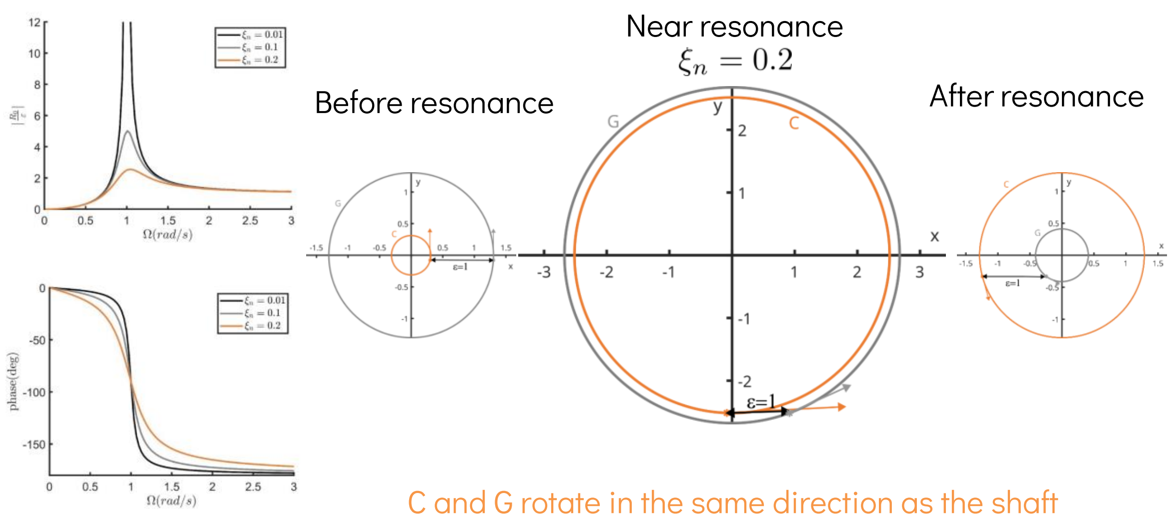
Interpretation of harmonic response



C and G rotate in the same direction as the shaft

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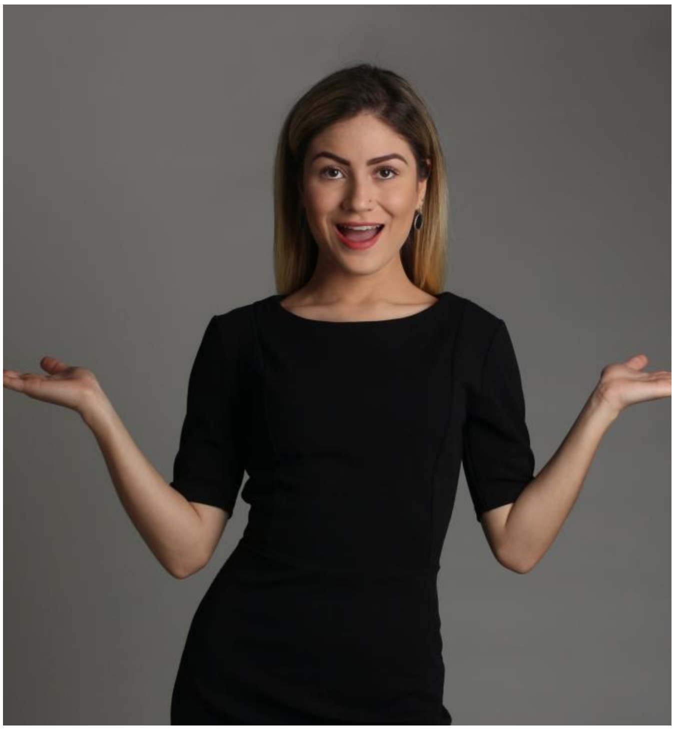
SUMMARY



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Jefcott rotor summary

- Rotation speed is critical when it corresponds to a resonance
- Without damping: the resonances do not depend on the rotation frequency
- With rotating damping: resonances depend on rotation and one pole becomes unstable for high rotation speed
- For the unbalance response, it is only the non-rotating damping which limits the amplitude at resonance

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