MECA-H303 TP2

Lagrange dynamics

Pendulum with mass spring

For the following mechanism, we ask you to:

1. Write the Lagrangien
2. Write the equations of motion

Additional details for the exercise:

- The inertias are given with respect to the centre of gravity of the related body
- The free length of the spring is at the position $x = 0$
- Consider $x << l$

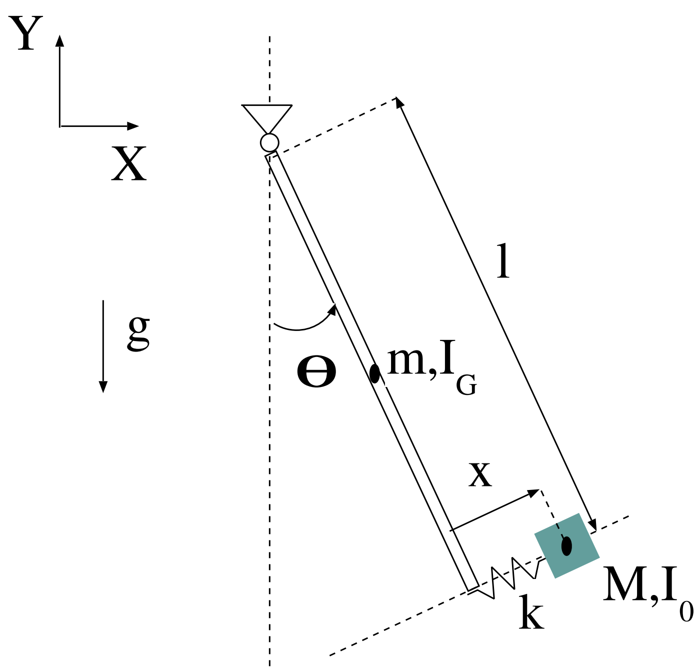


Figure 1: pendulum mass spring

$\boxed{1}$

$$T = \frac{1}{2} m \left(\frac{l}{2}\dot{\theta}\right)^2 + \frac{1}{2} I_G \dot{\theta}^2$$: kinetic energy of bar

$$+ \frac{1}{2} M v_M^2 + \frac{1}{2} I_0 \dot{\theta}^2$$: kinetic energy of mass spring

it's related only to the rotation of the body!

$$v_M = (l\dot{\theta} + \dot{x})^2 = (l\dot{\theta})^2 + \dot{x}^2 + 2l\dot{\theta}\dot{x}$$

not neglectible because it's the velocity

$$\Leftrightarrow T = \frac{1}{2}\left(M + \frac{m}{4}\right)l^2\dot{\theta}^2 + \frac{1}{2}(I_G + I_0)\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 + Ml\dot{\theta}\dot{x}$$

general formula $\boxed{V = mgh}$ with reference at the top hinge

$$V = mg\left(-\frac{l}{2}\cos\theta\right)$$: potential of bar

$$Mg\left(-l\cos\theta \cancel{+ x\sin\theta}\right)$$: potential of mass M

neglectible

$$+ \frac{1}{2}kx^2$$: potential from spring

always positive

$$L = T - V$$

$$= \frac{1}{2}\left(\left(M + \frac{m}{4}\right)l^2 + I_G + I_0\right)\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 + Ml\dot{\theta}\dot{x}$$

$$+ \left(mg\frac{l}{2} + Mgl\right)\cos\theta - \frac{1}{2}kx^2$$

$\boxed{2}$

$$\frac{d}{dt}\frac{dL}{d\dot{q}_i} - \frac{dL}{dq_i} = 0$$

<u>variable x</u>

$$\frac{dL}{dx} = -kx$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) = \frac{d}{dt}\left(M\dot{x} + Ml\dot{\theta}\right) = M\ddot{x} + Ml\ddot{\theta}$$

$$\Rightarrow \quad M\ddot{x} + Ml\ddot{\theta} + kx = 0$$

<u>variable θ</u>

$$\frac{dL}{d\theta} = -\left(mg\frac{l}{2} + Mgl\right)\sin\theta$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}}\right) = \frac{d}{dt}\left(\left(\left(M+\frac{m}{4}\right)l^2 + I_G + I_o\right)\dot{\theta} + Ml\dot{x}\right)$$

$$= \left(Ml^2 + \frac{m}{4}l^2 + I_G + I_o\right)\ddot{\theta} + Ml\ddot{x}$$

$$\Rightarrow \left(Ml^2 + \frac{m}{4}l^2 + I_G + I_o\right)\ddot{\theta} + Ml\ddot{x} + \left(\frac{m}{2} + M\right)gl\sin\theta = 0$$

Double pendulum inverted on a cart

For the following mechanism, we ask you to:

1. Write the Lagrangien
2. Write the equations of motion

Additional details for the exercise:

- The mass of all bars is negligible
- Black circles attached to the bars have a mass m

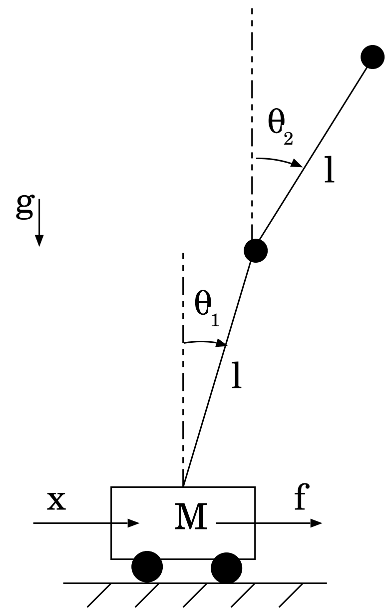
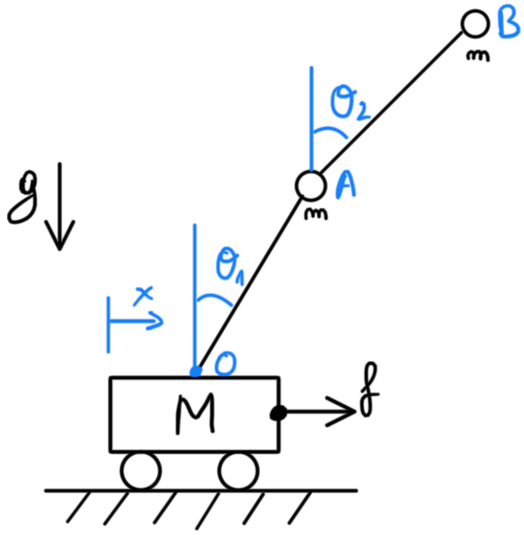


Figure 2: double pendulum inverted on a cart



$\boxed{1}$

$$L = T - V$$

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m v_A^2 + \frac{1}{2} m v_B^2$$

$$v_A = \frac{d}{dt}\left(\bar{x} + \overline{OA}\right) = \frac{d}{dt}\left(\left(x + l \sin\theta_1\right) \bar{1}_x + l \cos\theta_1 \bar{1}_y\right)$$

$$= \left(\dot{x} + l \dot{\theta}_1 \cos\theta_1\right) \bar{1}_x - l \dot{\theta}_1 \sin\theta_1 \bar{1}_y$$

$$v_A^2 = \dot{x}^2 + 2 l \dot{x} \dot{\theta}_1 \cos\theta_1 + \left(l^2 \dot{\theta}_1^2 \cos^2\theta_1 + l^2 \dot{\theta}_1^2 \sin^2\theta_1\right) \rightarrow \cos^2\theta + \sin^2\theta = 1$$

$$= \dot{x}^2 + 2 l \dot{x} \dot{\theta}_1 \cos\theta_1 + l^2 \dot{\theta}_1^2$$

$$v_B = \frac{d}{dt}\left(\bar{x} + \overline{OB}\right) = \frac{d}{dt}\left(\left(x + l \sin\theta_1 + l \sin\theta_2\right) \bar{1}_x + \left(l \cos\theta_1 + l \cos\theta_2\right) \bar{1}_y\right)$$

$$= \left(\dot{x} + l\left(\dot{\theta}_1 \cos\theta_1 + \dot{\theta}_2 \cos\theta_2\right)\right) \bar{1}_x - l\left(\dot{\theta}_1 \sin\theta_1 + \dot{\theta}_2 \sin\theta_2\right) \bar{1}_y$$

$$v_B^2 = \dot{x}^2 + 2 \dot{x} l \left(\dot{\theta}_1 \cos\theta_1 + \dot{\theta}_2 \cos\theta_2\right) + l^2\left(\dot{\theta}_1^2 \cos^2\theta_1 + 2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 + \dot{\theta}_2^2 \cos^2\theta_2\right)$$

$$+ l^2\left(\dot{\theta}_1^2 \sin^2\theta_1 + 2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2 + \dot{\theta}_2^2 \sin^2\theta_2\right)$$

$$v_B^2 = \dot{x}^2 + 2\dot{x}\, l\left(\dot{\theta}_1 \cos\theta_1 + \dot{\theta}_2 \cos\theta_2\right)$$
$$+ l^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \underbrace{2\dot{\theta}_1\dot{\theta}_2 \cos\theta_1 \cos\theta_2 + 2\dot{\theta}_1\dot{\theta}_2 \sin\theta_1 \sin\theta_2}\right)$$
$$\hookrightarrow = 2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)$$ choosed this one
$$\hookrightarrow = 2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$T = \frac{1}{2} M \dot{x}^2 + m\dot{x}^2 + 2ml\dot{x}\dot{\theta}_1 \cos\theta_1 + ml^2\dot{\theta}_1^2 + \frac{1}{2} m l^2 \dot{\theta}_2^2$$
$$+ m\dot{x} l \dot{\theta}_2 \cos\theta_2 + m l^2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$V = mgl \cos\theta_1 + mgl(\cos\theta_1 + \cos\theta_2)$$
$$= mgl(2\cos\theta_1 + \cos\theta_2)$$

$$L = T - V$$

$\boxed{2}$ $$\frac{d}{dt}\frac{dL}{d\dot{q}_i} - \frac{dL}{dq_i} = Q_i$$

<u>variable x</u> $\qquad Q_i = f \qquad \dfrac{dL}{dx} = 0$

$$\frac{dL}{d\dot{x}} = M\dot{x} + 2m\dot{x} + 2ml\dot{\theta}_1 \cos\theta_1 + ml\dot{\theta}_2 \cos\theta_2$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) = M\ddot{x} + 2m\ddot{x} + 2ml\ddot{\theta}_1 \cos\theta_1 - 2ml\dot{\theta}_1^2 \sin\theta_1$$
$$+ ml\ddot{\theta}_2 \cos\theta_2 - ml\dot{\theta}_2^2 \sin\theta_2$$

$$\frac{d}{dt}\left(\frac{dL}{dx}\right) - \frac{dL}{dx} = f$$

$$(M + 2m)\ddot{x} + ml\left(2\ddot{\theta}_1 \cos\theta_1 + \ddot{\theta}_2 \cos\theta_2\right) - ml\left(2\dot{\theta}_1^2 \sin\theta_1 + \dot{\theta}_2^2 \sin\theta_2\right) = f$$

<u>variable θ_1</u>

$$Q_i = 0$$

$$\frac{\partial L}{\partial \theta_1} = -2\, m\, l\, \dot{x}\, \dot{\theta}_1 \sin\theta_1 - m\, l^2 \dot{\theta}_1 \dot{\theta}_2 \,\text{sen}\,(\theta_1 - \theta_2) + 2 m g l \sin\theta_1$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = 2 m\, l\, \dot{x} \cos\theta_1 + 2 m\, l^2 \dot{\theta}_1 + m\, l^2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = 2 m\, l \left(\ddot{x} \cos\theta_1 - \dot{x}\, \dot{\theta}_1 \sin\theta_1\right) + 2 m\, l^2 \ddot{\theta}_1$$
$$+ m\, l^2 \left(\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2 \,\text{sin}\,(\theta_1 - \theta_2) . (\dot{\theta}_1 - \dot{\theta}_2)\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\Leftrightarrow 2 m\, l\, \ddot{x} \cos\theta_1 - 2 m g l \sin\theta_1 + 2 m\, l^2 \ddot{\theta}_1$$
$$+ m\, l^2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m\, l^2 \dot{\theta}_2^{\,2} \,\text{sen}\,(\theta_1 - \theta_2) = 0$$

variable θ_2

$$Q_i = 0$$

$$\frac{\partial(\theta_1-\theta_2)}{d\theta_2} = -1$$

$$\frac{\partial L}{d\theta_2} = -m\dot{x}\,l\,\dot{\theta}_2 \sin\theta_2 - m l^2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2)\cdot(-1) + mgl\sin\theta_2$$

$$\frac{dL}{d\dot{\theta}_2} = m l^2 \dot{\theta}_2 + m\dot{x}\,l\cos\theta_2 + m l^2 \dot{\theta}_1 \cos(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_2}\right) = m l^2 \ddot{\theta}_2 + m l\left(\ddot{x}\cos\theta_2 - \dot{x}\dot{\theta}_2\sin\theta_2\right)$$
$$+ m l^2\left(\ddot{\theta}_1\cos(\theta_1-\theta_2) - \dot{\theta}_1 \sin(\theta_1-\theta_2)\cdot(\dot{\theta}_1-\dot{\theta}_2)\right)$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_2}\right) - \frac{dL}{d\theta_2} = 0$$

$$\Leftrightarrow \quad -mgl\sin\theta_2 + m l^2\left(\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1-\theta_2)\right) + m l\ddot{x}\cos\theta_2$$
$$- m l^2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) = 0$$