MECA-H303 TP3

MDOF

Double shaft in torsion

For the following system we ask you to:

1. Write the equations of motion in analytical and matrix form
2. Give the expression of the mass matrix and stiffness matrix of this system
3. Give the analytical expressions of natural frequency and eigen modes of the system.

Additional details for the exercise:

- The shafts have only a torsional rigidity k (in $[N/rad]$)
- The discs have an inertia I
- A torque C is applied to the first disc

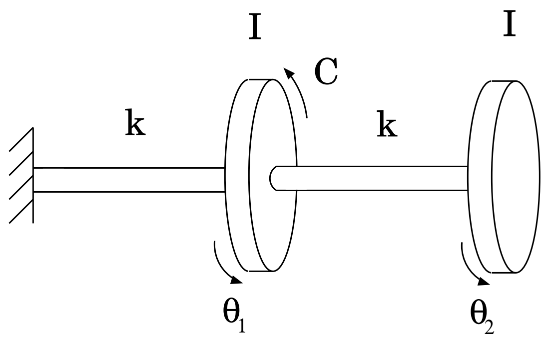


Figure 1: double shaft in torsion

1

Angular momentum formula:

$$\frac{d\bar{M}_A}{dt} = m v_G \times v_A + \sum C_{ext}$$

$$\iff I\ddot{\theta} = \sum C_{ext} \quad \text{at each disc}$$

$$I\ddot{\theta}_1 = \sum C_{ext}$$
$$= -k\theta_1 + C + k(\theta_2 - \theta_1)$$

$$I\ddot{\theta}_2 = \sum C_{ext}$$
$$= -k(\theta_2 - \theta_1)$$

$$\iff \begin{cases} I\ddot{\theta}_1 + 2k\theta_1 - k\theta_2 = C \\ I\ddot{\theta}_2 - k\theta_1 + k\theta_2 = 0 \end{cases}$$

$$\iff \underbrace{\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}}_{\substack{\text{Mass matrix} \\ M}} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \underbrace{\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}}_{\substack{\text{Stiffness matrix} \\ K}} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} C \\ 0 \end{bmatrix}$$

↳ always positive

2

$\boxed{3}$

<u>natural frequencies</u> (eigen values)

$$\det(K - \omega^2 M) = 0 \iff \omega_i^2 = ?$$

$$\det\left(\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} - \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\right) = 0$$

$$\iff \det\left(\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} - \begin{bmatrix} I\omega^2 & 0 \\ 0 & I\omega^2 \end{bmatrix}\right) = 0$$

$$\iff \det\left(\begin{bmatrix} 2k - \omega^2 I & -k \\ -k & k - \omega^2 I \end{bmatrix}\right) = 0$$

$$\iff (2k - \omega^2 I)\cdot(k - \omega^2 I) - k^2 = 0$$

$$\iff 2k^2 - k\omega^2 I - 2k\omega^2 I + \omega^4 I^2 - k^2 = 0$$

$$\iff \omega^4 I^2 - \omega^2 3kI + k^2 = 0 \qquad (\text{assuming } \lambda = \omega^2)$$

$$\iff \lambda^2 I^2 - \lambda 3kI + k^2 = 0$$

$$\sqrt{\Delta} = (3kI)^2 - 4.\, I^2 k^2 = 5k^2 I^2 \qquad \lambda_{1,2} = \frac{+3kI \pm \sqrt{5}\,kI}{2I^2}$$

$$\iff \begin{cases} \omega_1^2 = \dfrac{(3+\sqrt{5})}{2}\,\dfrac{k}{I} \\[2ex] \omega_2^2 = \dfrac{3-\sqrt{5}}{2}\,\dfrac{k}{I} \end{cases} \rightarrow \text{natural frequencies or eigen values in } [\mathrm{rad}^2/\mathrm{s}^2]$$

$$b = \frac{3+\sqrt{5}}{2} \simeq 2{,}618 \qquad c = \frac{3-\sqrt{5}}{2} \simeq 0{,}382$$

mode shapes (eigen vectors)

$$\left(K - \omega^2 M\right)\Psi = \{0\} \iff \Psi_i = ?$$

$$\iff \left(\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} - \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} (2k - \omega^2 I)\varphi_1 - k\varphi_2 = 0 & (1) \\ -k\varphi_1 + (k - \omega^2 I)\varphi_2 = 0 & (2) \end{cases}$$

$$\omega^2 = a\frac{k}{I}$$

where $a = \begin{cases} b & \text{for } \omega_1^2 \\ c & \text{for } \omega_2^2 \end{cases}$

$$\iff \begin{cases} (2-a)\varphi_1 - \varphi_2 = 0 \\ -\varphi_1 + (1-a)\varphi_2 = 0 \end{cases}$$

$$\iff \begin{cases} \varphi_1 = \varphi_2 \dfrac{1}{2-a} \\ \varphi_1 = \varphi_2(1-a) \end{cases} \longrightarrow$$

Identity relation that should be true for $a = \begin{cases} b = 2,618 \\ c = 0,382 \end{cases}$

$$\frac{1}{2-a} = 1 - a$$

$$\omega_1^2 = 2,618\frac{k}{I}$$

we choose $\varphi_2 = 2 - b = -0,618$

$\hookrightarrow \varphi_1 = 1$

$$\Psi_1 = \begin{pmatrix} 1 \\ -0,618 \end{pmatrix}$$

$$\omega_2^2 = 0,382\frac{k}{I}$$

we choose $\varphi_2 = 2 - c = 1,618$

$\hookrightarrow \varphi_1 = 1$

$$\Psi_2 = \begin{pmatrix} 1 \\ 1,618 \end{pmatrix}$$

MDOF & Isolation

Car suspension

The following mass/spring/damper system represents the suspension of one wheel in a car. For this system we ask you to:

1. Write the equations of motion of the suspension system in analytical and matrix form
2. Give the analytical expressions of the natural frequencies and eigen modes of the system. Compute them for the following numerical values:
   - $m_s = 240kg$
   - $m_{us} = 36kg$
   - $k_t = 160kN/m$
   - $k = 16kN/m$
3. Compare the natural frequency of the full system that you just calculated with the one of the un-coupled systems (fixed wheel & wheel alone) and comment on the differences between the un-coupled and coupled systems
4. Give the expression of the **static** displacement resulting from the application of an unit displacement w.
5. In the frequency domain, write the transfer function between the velocity of the road $\dot{w}$ (input) and the acceleration of the suspended mass $\ddot{x}$ (output).

Additional details for the exercise:

- m_s is the suspended mass of the car on a wheel, corresponding to 1/4th of the full mass of the car
- m_{us} is the un-suspended mass of the car (the wheel)
- c is the damping provided by the suspension between the wheel and the car
- w is the base displacement imposed by the road

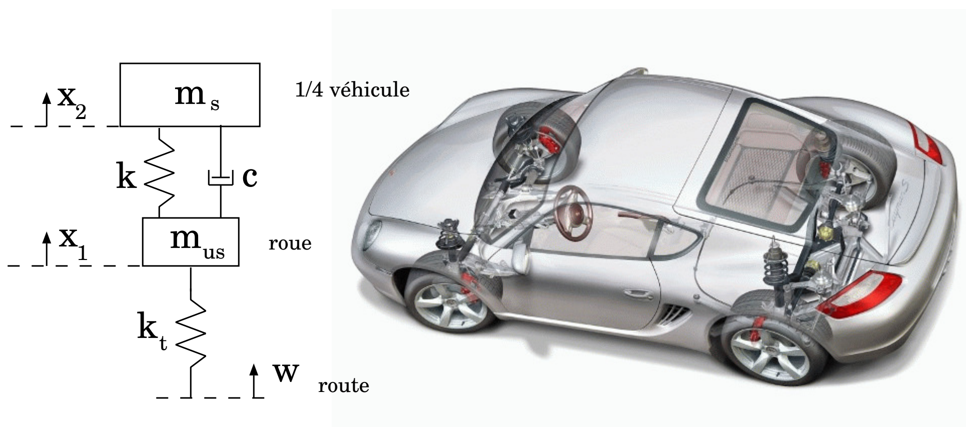


Figure 2: car suspension

$\boxed{1}$ Equation of motion

Newton's equation of linear momentum $m\ddot{x} = \sum F_x$

<u>For mass m</u>

$$m_s \ddot{x}_2 = -k(x_2 - x_1) - c(\dot{x}_2 - \dot{x}_1)$$

$$\Leftrightarrow m_s \ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 - kx_1 + kx_2 = 0$$

<u>For mass m_{us}</u>

$$m_{us} \ddot{x}_1 = -k_t(x_1 - w) - k(x_1 - x_2) - c(\dot{x}_1 - \dot{x}_2)$$

$$\Leftrightarrow m_{us} \ddot{x}_1 + c\dot{x}_1 - c\dot{x}_2 + (k_t + k)x_1 - kx_2 = k_t w$$

<u>In matrix form</u>

$$\begin{bmatrix} m_{us} & 0 \\ 0 & m_s \end{bmatrix} \ddot{X} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \dot{X} + \begin{bmatrix} k_t + k & -k \\ -k & k \end{bmatrix} X = \begin{bmatrix} k_t w \\ 0 \end{bmatrix}$$

$$M \quad \ddot{X} \; + \quad C \quad \dot{X} \; + \quad K X \quad = \quad F$$

where $\ddot{X} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}$; $\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\boxed{2}$

Eigen values (natural frequencies)

$$\det(K - \omega^2 M) = 0$$

$$\Leftrightarrow \det\begin{pmatrix} (k_t+k) - \omega^2 m_{us} & -k \\ -k & k - \omega^2 m_s \end{pmatrix} = 0$$

$$\Leftrightarrow (k_t + k - \omega^2 m_{us})(k - \omega^2 m_s) - k^2 = 0$$

$$\Leftrightarrow k_t k - \omega^2 k_t m_s + \cancel{k^2} - k\omega^2 m_s - k\omega^2 m_{us} + \omega^4 m_s m_{us} - \cancel{k^2} = 0$$

$$\Leftrightarrow \omega^4 m_s m_{us} - \omega^2\left((k_t + k) m_s + k m_{us}\right) + k_t k = 0$$

$$\sqrt{\Delta} = \sqrt{((k_t+k) m_s + k m_{us})^2 - 4 m_s m_{us} k_t k}$$

$$= \sqrt{((160000+16000).240 + 16000.36)^2 - 4.240.36.160000.16000}$$

$$= 41\ 770\ 040$$

$$\omega^2 = \frac{k_t m + k m + k m_{us} \pm \sqrt{\Delta}}{2.m\, m_{us}} = \frac{160000.240 + 16000.240 + 16000.36 \pm 42{,}92.10^6}{2.240.360}$$

$$\omega_1^2 = 4\,895\ (rad/s)^2 \qquad \Rightarrow \qquad \omega_1 = 69{,}96\ rad/s \Leftrightarrow f_1 = 11{,}13\ Hz$$

$$\omega_2^2 = 60{,}53\ (rad/s)^2 \qquad \qquad \omega_2 = 7{,}78\ rad/s \Leftrightarrow f_2 = 1{,}24\ Hz$$

Eigen vectors (Mode shapes)

$$(K - \omega^2 M)\Psi = 0$$

$$\begin{bmatrix} (k_t + k) - \omega^2 m_{us} & -k \\ -k & k - \omega^2 m_s \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} k_t + k - \omega^2 m_{us} \varphi_1 - k \varphi_2 = 0 \\ -k \varphi_1 + (k - \omega^2 m_s) \varphi_2 = 0 \end{cases}$$

$$\begin{cases} \varphi_1 = \dfrac{k}{k_t + k - \omega^2 m_{us}} \varphi_2 \\ \varphi_1 = \dfrac{k - \omega^2 m_s}{k} \varphi_2 \end{cases} \rightarrow$$

$$\boxed{\text{Identity relation valid for } \omega^2 = \begin{cases} \omega_1^2 \\ \omega_2^2 \end{cases} \quad \frac{k - \omega^2 m_s}{k} = \frac{k}{k_t + k - \omega^2 m_{us}}}$$

ω_1^2

we choose $\varphi_1 = \dfrac{k}{k_t + k - \omega_1^2 m_{us}}$

$\Rightarrow \varphi_2 = 1$

$$\Psi_1 = \begin{pmatrix} -72{,}73 \\ 1 \end{pmatrix}$$

wheel moves a lot

unsuspended mass resonance

ω_2^2

we choose $\varphi_2 = \dfrac{k_t + k - \omega_2^2 m_{us}}{k}$

$\varphi_1 = 1$

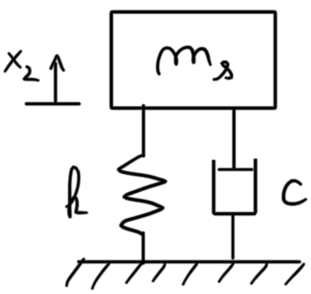
$$\Psi_2 = \begin{pmatrix} 1 \\ 10{,}86 \end{pmatrix}$$

car moves a lot

suspended mass resonance

$\boxed{3}$

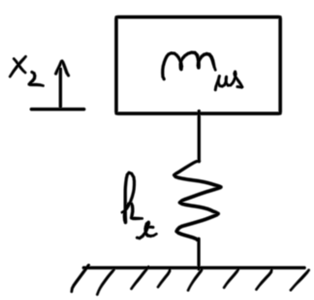
fixed wheel



$$\omega_s = \sqrt{\frac{k}{m_s}} = 66,67 \text{ rad/s}$$

$$f = 10,61 \text{ Hz}$$

wheel alone



$$\omega_{us} = \sqrt{\frac{k_t}{m_{us}}} = 8,16 \text{ rad/s}$$

$$f_{us} = 1,30 \text{ Hz}$$

Comparison

The un-coupled system has natural frequencies close to the coupled system but, there is a non-negligeable change of their natural frequencies when they are coupled. This is due to the fact that their natural frequencies are close.

The frequencies of the coupled system also have a tendency to be further apart compared to the un-coupled system.

$\boxed{4}$

$$M\ddot{X} + C\ddot{X} + KX = F \xrightarrow{\text{In statics}} KX = F$$

$$\begin{bmatrix} k+k_t & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_t w \\ 0 \end{bmatrix}$$

$$\begin{cases} (k+k_t)x_1 - k x_2 = k_t w \\ -k x_1 + k x_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \cancel{k} x_1 = \cancel{k} x_2 \\ \cancel{k x_1} + \cancel{k_t} x_1 - \cancel{k x_1} = \cancel{k_t} w \end{cases}$$

$$\Leftrightarrow \begin{cases} x_2 = x_1 \\ x_1 = w \end{cases}$$

$$\Leftrightarrow x_2 = w \qquad w = 1 \rightarrow \begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases}$$

$\boxed{5}$

$$T = \frac{-\omega^2 X_2}{i\omega W} = \frac{\ddot{X}_2}{\dot{W}}$$

$$\begin{cases} m_{us}\ddot{x}_1 + c\dot{x}_1 - c\dot{x}_2 + (k_t + k)\, x_1 - k x_2 = k_t w \\ m_s \ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 \quad - k x_1 \quad + k x_2 = 0 \end{cases}$$

In frequency domain

$$\boxed{\begin{aligned} x_1 &= X_1 e^{i\omega t} \\ x_2 &= X_2 e^{i\omega t} \\ w &= W e^{i\omega t} \end{aligned}}$$

$$\begin{cases} -\omega^2 m_{us} X_1 + i\omega c X_1 - i\omega c X_2 + (k_t + k) X_1 - k X_2 = k_t W \\ -\omega^2 m_s X_2 - i\omega c X_1 + i\omega c X_2 - k X_1 + k X_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(-\omega^2 m_{us} + i\omega c + (k_t + k)\right) X_1 - (i\omega c + k) X_2 = k_t W \\ -(i\omega c + k) \quad X_1 + (-\omega^2 m_s + i\omega c + k) X_2 \quad = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\omega^2 m_{us} + i\omega c + k_t + k) \dfrac{-\omega^2 m_s + i\omega c + k}{i\omega c + k} X_2 - (i\omega c + k) X_2 = k_t W \\ X_1 = \dfrac{-\omega^2 m_s + i\omega c + k}{i\omega c + k} X_2 \end{cases}$$

$$\rightarrow \quad \frac{(-\omega^2 m_{us} + i\omega c + k_t + k)(-\omega^2 m_s + i\omega c + k) - (i\omega c + k)^2}{i\omega c + k} X_2 = k_t W$$

$$\Leftrightarrow \frac{X_2}{W} = \frac{k_t(i\omega c + k)}{(-\omega^2 m_{us} + i\omega c + k_t + k)(-\omega^2 m_s + i\omega c + k) - (i\omega c + k)^2}$$

$$\Leftrightarrow \frac{\ddot{X}_2}{\dot{W}} = \frac{-\omega}{i} \frac{k_t(i\omega c + k)}{(-\omega^2 m_{us} + i\omega c + k_t + k)(-\omega^2 m_s + i\omega c + k) - (i\omega c + k)^2}$$