# MECA H 303:
# Kinematics and dynamics of machines

## Partim: Dynamics and vibrations
## Introduction

Exercise session 1

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# P1.1

The planar mechanism under study in this exercise is shown in Fig. 1.

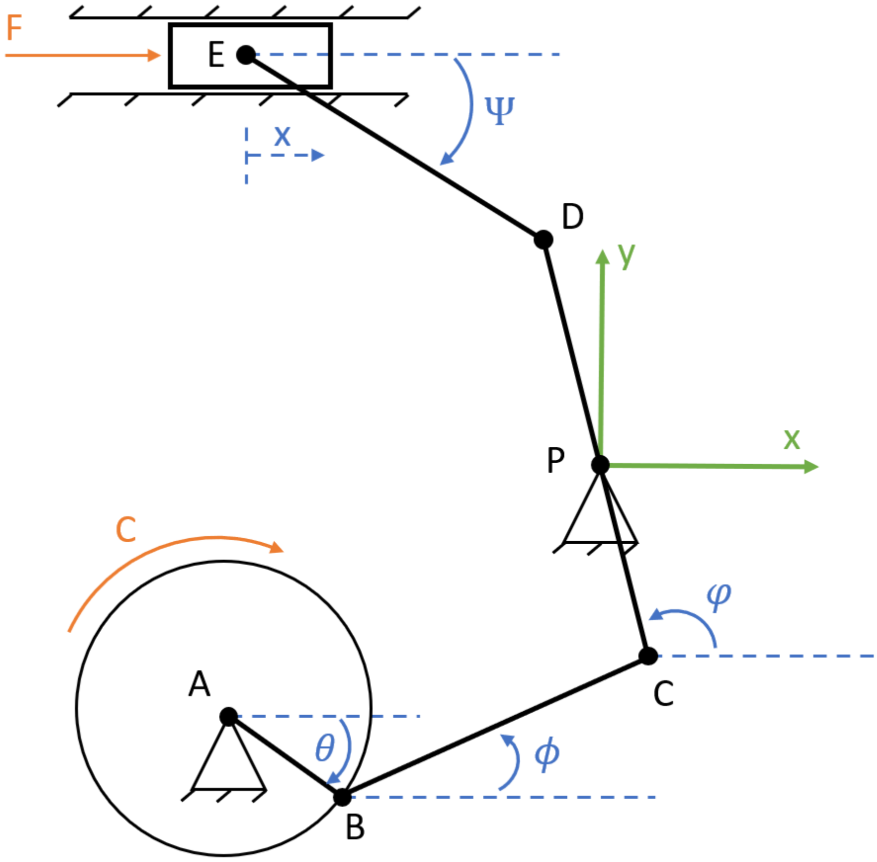



Figure 1: Planar mechanism

### Number of degrees of freedom

The number of degrees of freedom of a mechanism is given by

$$N = \sum_{i=1}^{l} \nu_i - 6 \cdot b \tag{1}$$

However, because the mechanism is planar, only three degrees of freedom are allowed for each body, so that the equation to use is

$$N = \sum_{i=1}^{l} \nu_i - 3 \cdot b \tag{2}$$

The $\nu_i$'s coefficients are the number of degrees of freedom allowed by each joints. In this case, it is:

$$\sum_{i=1}^{6} \nu_i = \underbrace{2}_{\text{E}} + 1 + 1 + 1 + 1 + 1 = 7 \tag{3}$$

$b$ is the number of closed loop, it is given by:

$$b = l - n = 6 - 4 = 2 \tag{4}$$

where $l$ is the number of joints and $n$ the number of bodies.

In the end, it leads us to

$$N = 7 - 3 \cdot 2 = 1 \tag{5}$$

**Constraints equations**

The mechanism in Fig. 1 can be described using 5 coordinates. They are here chosen to be the relative coordinates between each bodies:

$$\mathbf{q} = \{x, \Psi, \varphi, \phi, \theta\} \tag{6}$$

But because our system has only 1 degree of freedom, there must exist 4 constraint equations

$$p = q - N = 5 - 1 = 4 \tag{7}$$

As a general rule, the constraint equations are obtained by expressing mathematically the restriction of the relative motion between two bodies. The mathematical formalization is easier when expressing these relations in an absolute reference frame. A sensible choice is to chose a fixed joints of the mechanism, for example joints P in Fig. 1, and to express the motions of the extremities of the mechanisms with respect ot that point. The reference frame is represented in green in the schematic. The four constraints can be obtained:

$$\Phi_i \begin{cases} EP_x = x + ED\cos\Psi + DP\cos\varphi \\ EP_y = DP\sin\varphi + ED\sin\Psi \\ AP_x = AB\cos\theta + BC\cos\phi + CP\cos\varphi \\ AP_y = -AB\sin\theta + BC\sin\phi + CP\sin\varphi \end{cases} \tag{8}$$

**Ratio F/C**

The mechanism in fig. 1 can be used to convert a linear force F to a torque C, or conversely. In order to determine the ratio of the force to the applied torque, the principle of virtual work can be used:

$$\begin{aligned} \partial\tau &= F\delta x + C\delta\theta = 0 \\ \Rightarrow F/C &= \frac{-\delta\theta}{\delta x} \end{aligned} \tag{9}$$

The time evolution of $x$ and $\theta$ can be obtained solving the constraints equations. The coordinate partitioning method is used.

$$d\Phi = \sum_i \frac{\partial\mathbf{\Phi}}{\partial q_i}\partial q_i = 0 \tag{10}$$

The partial derivation of each equations in Eq. 8 leads to

$$\begin{cases} 0 = \delta x - ED\sin\Psi\delta\Psi - DP\sin\varphi\delta\varphi \\ 0 = DP\cos\varphi\delta\varphi + ED\cos\Psi\delta\Psi \\ 0 = -AB\sin\theta\delta\theta - BC\sin\phi\delta\phi - CP\sin\varphi\delta\varphi \\ 0 = -AB\cos\theta\delta\theta + BC\cos\phi\delta\phi + CP\cos\varphi\delta\varphi \end{cases} \tag{11}$$

Doing some algebra, one can derive that, from the fourth equation in Eq. 11,

$$\delta\phi = \frac{AB\cos\theta\delta\theta - CP\cos\varphi\delta\varphi}{BC\cos\phi} \tag{12}$$

Substituting in the third one and solving for $\delta\varphi$:

$$\begin{aligned} \delta\varphi &= \frac{(AB\sin\theta + AB\tan\phi\cos\theta)\delta\theta}{CP\tan\phi\cos\varphi - CP\sin\varphi} \\ &= M\delta\theta \end{aligned} \tag{13}$$

The second equation can be directly solved for $\delta\Psi$:

$$\delta\Psi = \frac{-DP\cos\varphi\delta\varphi}{ED\cos\Psi} \tag{14}$$

Substituting $\delta\Psi$ and $\delta\phi$ with the above equations in the first equation of Eq. 11, one obtains

$$\delta x + DP\tan\Psi\cos\varphi\delta\varphi - DP\sin\varphi\delta\varphi = \delta x + DP\tan\Psi\cos\varphi M\delta\theta - DP\sin\varphi M\delta\theta = 0 \tag{15}$$

So that, in the end:

$$\frac{F}{C} = \frac{-\delta\theta}{\delta x} = \frac{PC\tan\phi\cos\Psi - PC\sin\varphi}{(AB\sin\theta + AB\tan\phi\cos\theta)(DP\tan\varphi\cos\varphi - DP\sin\varphi)} \tag{16}$$

## P1.2

Let us consider the system made of two shafts connected with spur gears shown in Fig. 2. The torsional stiffness of the shaft are given as $GJ$, the mass moment of Inertia are $I_A$ and $I_B$, and the ratios of the spur gear radii is $R_A/R_B = n$.

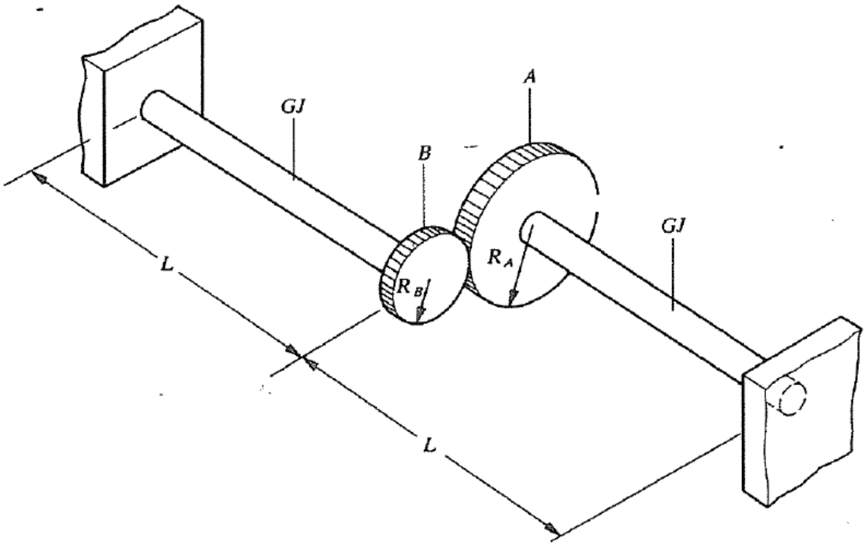



Figure 2: Spur gear system

**Differential equation of the system**

Each shaft can be described using their own absolute coordinates, let us call $\theta_A$ and $\theta_B$. But because of the spur gear, the system has only 1 degree of freedom, and there must exist $p = q - N = 2 - 1$ constraint equation. The constraint can be expressed by the fact that the two gears makes contact without sliding, i.e. the linear speed at the contact point of the two shaft is the same for the two shaft.

$$\begin{aligned} R_A\dot{\theta}_A &= R_B\dot{\theta}_B \\ \Rightarrow \dot{\theta}_B &= \frac{-R_A}{R_B}\dot{\theta}_A = -n\dot{\theta}_A \end{aligned} \tag{17}$$

After integration,

$$\theta_B = -n\theta_A \tag{18}$$

The equation of motion of the system can be obtained the analytical way using Lagrange's equation.

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{q}}\right) - \frac{d\mathcal{L}}{dq} = 0 \tag{19}$$

$q$ is the degree of freedom of the system, in this case $\theta_A$. $\mathcal{L} = \mathcal{T} - \mathcal{V}$ is the Lagrangian of the system. $\mathcal{T}$ corresponds to the kinetic energy stored in the rotation motion of the two shafts:

$$\mathcal{T} = \frac{1}{2}I_A\dot{\theta}_A^2 + \frac{1}{2}I_B\dot{\theta}_B^2 \tag{20}$$

$\mathcal{V}$ is the potential energy stored in the elastic deformation of the two shafts:

$$\mathcal{V} = \frac{1}{2}k\theta_A^2 + \frac{1}{2}k\theta_B^2 \tag{21}$$

where $k = \frac{GJ}{L}$.

The Lagrangian, expressed using the $\theta_A$ degree of freedom, is:

$$\mathcal{L} = \frac{1}{2}\left[\left(I_A + I_B r^2\right)\dot{\theta}_A^2 - \left((1+r^2)k\theta_A^2\right)\right] \tag{22}$$

Applying the Lagrange equation Eq. 19, it yields

$$\left(I_A + r^2 I_B\right)\ddot{\theta}_A + \left(1 + r^2\right)k\theta_A = 0 \tag{23}$$

**Natural frequency of the system**

The dynamic of the system is studied by turning the equation of motion from the time domain to the Laplace domain. This is done by assuming an harmonic motion of the system, i.e. $\theta_A = Ae^{st}$, where $s$ is the Laplace variable ($s = i\omega$). It leads to:

$$\begin{aligned}&\left(I_A + r^2 I_B\right)s^2 Ae^{st} + \left(1 + r^2\right)kAe^{st} = 0\\&\left[\left(I_A + r^2 I_B\right)s^2 + k(1+r^2)\right]Ae^{st} = 0\end{aligned} \tag{24}$$

Solving this equation leads to the following solutions:

$$s_{1,2} = \pm i\omega_0 = \pm i\sqrt{\frac{k(1+r^2)}{I_A + I_B r^2}} \tag{25}$$

where $\omega_0$ is the natural frequency of the system.

# P1.3

In order to measure the natural frequency of a control tab, the device is attached to two springs, one of which being excited until the resonance frequency of the setup is reached ($\omega_r$). Based on that measurement of $\omega_r$ and on the parameters of the experimental setup, recover the natural frequency of the control tab $\omega_0^2 = k_T/I_0$.

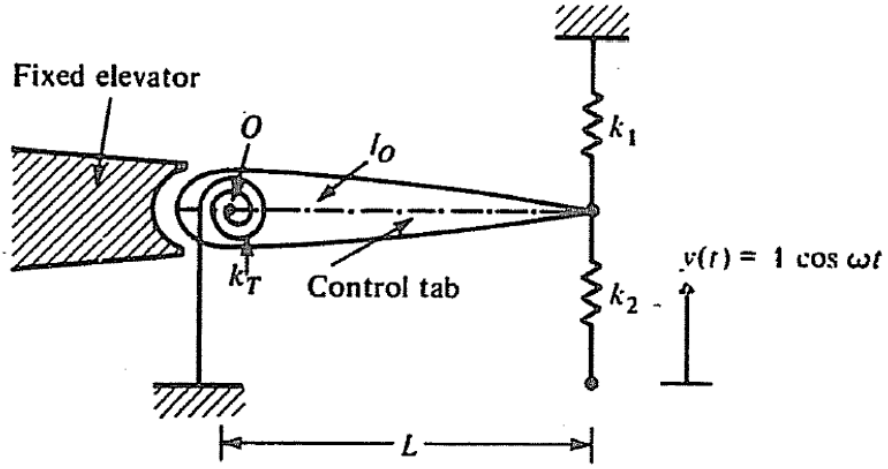



Figure 3: Control tab experimental setup

The equation of motion of the system is computed through the Lagrange equation. The Lagrangian is computed as

$$\mathcal{L} = \mathcal{T} - \mathcal{V} \tag{26}$$

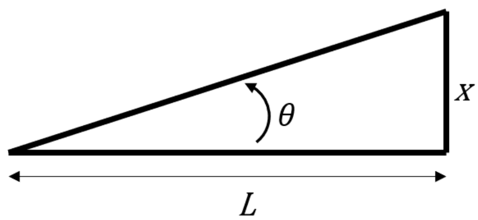
The kinetic energy of the control tab rotating about the $O$ point in Fig. 2 is

$$\mathcal{T} = \frac{1}{2} I_0 \dot{\theta}^2 \tag{27}$$

The potential energy stored in the torsional spring $k_T$ and the two linear spring $K_1$ and $k_2$ is

$$\mathcal{V} = \frac{1}{2} k_T \theta^2 + \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 (x - \underbrace{y}_{0})^2 \tag{28}$$

The linear $x$ displacement of the tip of the tab can be related to the rotation motion of the tab using a simple geometric relation, so that the potential energy:




$$\mathcal{V} = \frac{1}{2} k_T \theta^2 + \frac{1}{2} k_1 (\theta L)^2 + \frac{1}{2} k_2 (\theta L)^2 \tag{29}$$

So that the Lagrangian is:

$$\mathcal{L} = \frac{1}{2} \left[ I_0 \dot{\theta}^2 - \left( k_T + L^2 (k_1 + k_2) \right) \theta^2 \right] \tag{30}$$

The Lagrange equation is then applied to get the equation of motion of the system

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}}\right) - \frac{d\mathcal{L}}{d\theta} = 0 \Rightarrow I_0\ddot{\theta} + \left(k_T + L^2(k_1 + k_2)\right)\theta = 0 \tag{31}$$

Turning the equation to the Laplace domain considering $\theta = Ae^{st}$:

$$\left[I_0 s^2 + k_T + L^2(k_1 + k_2)\right] Ae^{st} = 0 \tag{32}$$

Solving the equation leads to the solutions

$$s_{1,2} = \pm i\omega_r = \pm i\sqrt{\frac{L^2(k_1 + k_2) + k_T}{I_0}} \tag{33}$$

Remembering that the resonance frequency of the control tab alone can be expressed as $\omega_0^2 = \frac{k_T}{I_0}$, the natural frequency of the experimental setup can be expressed as

$$\begin{aligned} \omega_r^2 &= \frac{k_T}{I_0} + \frac{L^2(k_1 + k_2)}{I_0} = \omega_0^2 + \frac{L^2(k_1 + k_2)}{I_0} \\ \Longleftrightarrow \omega_0 &= \sqrt{\omega_r^2 - \frac{L^2(k_1 + k_2)}{I_0}} \end{aligned} \tag{34}$$

which can be used to assess the natural frequency of the control tab, based on the experimental identification of $\omega_r$ and the characteristics of the setup.

## P1.4

A shaker with two counter rotating unbalanced masses is placed on a mass resting on a massless elastic floor. Calculate the magnitude of the response of the system as a function of the excitation frequency of the shaker.

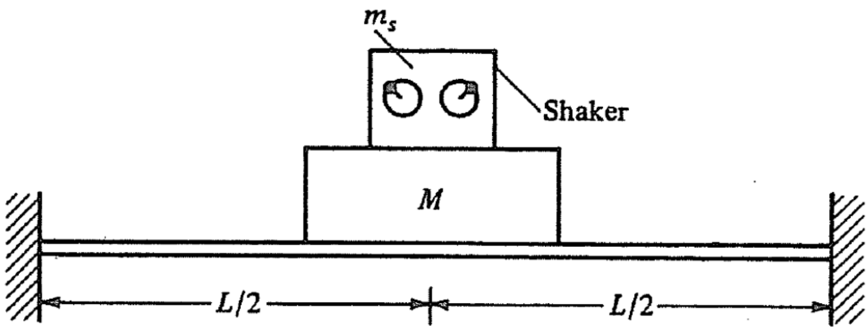



Figure 4: Shaker on an elastic floor.

The dynamic equation of this system can be assessed using the formalism of the Newton's law of motion:

$$\sum_i F_i = m_{total}\ddot{x} \tag{35}$$

where $x$ is the vertical motion of the mass at midspan, i.e. $L/2$. In order to ease the interpretation of the system, the Fig. 4 can be equivalently represented as

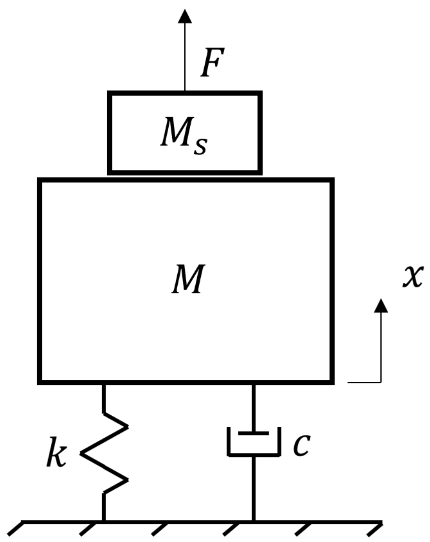



Figure 5: Simplified representation of the elastic, massless, beam.

The two counter rotating masses inside of the shaker are actuated at a rotation speed $\omega$. This generate a vertical harmonic force on the beam with an amplitude of $F = mL\omega^2 \sin(\omega t)$, where $m$ and $L$ are the mass and length of the rotating masses inside of the shaker and $\omega$ the rotation speed. The restoring force related to the elasticity of the beam, modelled by a stiffness $k$, is considered as $F_k = -kx$. A dashpot represents the dissipation phenomena in the system, whose force is proportional of velocity

$F_c = -c\dot{x}.$

This leads to the well-known equation of motion:

$$\begin{aligned} &(M + M_s + m)\ddot{x} = mL\omega^2 \sin(\omega t) - c\dot{x} - kx \\ \iff &(M + M_s + m)\ddot{x} + c\dot{x} + kx = mL\omega^2 \sin(\omega t) \end{aligned} \tag{36}$$

The excitation is harmonic and can be written as $F = mL\omega^2 F e^{j\omega t}$. The response is therefore necessarily harmonic, and can be written as $x = Xe^{j\omega t}$.

$$\left[(M + M_s + m)(j\omega)^2 + j\omega c + k\right] Xe^{j\omega t} = mL\omega^2 F e^{j\omega t} \tag{37}$$

So that the transfer function from the external force of the shaker $F$ to the response of the system $X$ is

$$\frac{X}{F} = \frac{mL\omega^2}{-(M + M_s + m)\omega^2 + j\omega c + k} \tag{38}$$

To get the magnitude of transfer function, we first need to get rid of the complex number appearing at the denominator. Recalling some principle of basic algebra, for an arbitrary complex number:

$$\frac{1}{a + jb} = \frac{a - jb}{(a + jb)(a - jb)} = \frac{a}{a^2 + b^2} + \frac{-jb}{a^2 + b^2} \tag{39}$$

And the magnitude of that complex number is

$$\left|\frac{1}{a + jb}\right| = \sqrt{\frac{a^2}{(a^2 + b^2)^2} + \frac{b^2}{(a^2 + b^2)^2}} = \sqrt{\frac{1}{a^2 + b^2}} \tag{40}$$

By direct analogy, we have $a \equiv (k - (M + M_s + m)\omega^2)$ and $b \equiv \omega c$. So that it comes:

$$\left|\frac{X}{F}\right| = mL\omega^2 \sqrt{\frac{1}{(k - (M + M_s + m)\omega^2)^2 + \omega^2 c^2}} \tag{41}$$

This transfer function can be re-written using variables which are more convenient to manipulate both in theory and in experiments. Making an analogy with a single degree of freedom system, one can define the natural frequency as $\omega_0 = \sqrt{k/m_{total}}$, and the damping coefficient $\xi = c/(2\omega_0 m_{total})$. So that, dividing Eq. 41 by $(M + M_s + m)$, on can be write:

$$\left|\frac{X}{F}\right| = \frac{mL\omega^2}{(M + M_s + m)} \sqrt{\frac{1}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2}} \tag{42}$$

On can further develop Eq. 42, by dividing by $\omega_0^2$, in order to obtain an even more standard notation:

$$\left|\frac{X}{F}\right| = \frac{mL}{M + M_s + m} \left(\frac{\omega}{\omega_0}\right)^2 \sqrt{\frac{1}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_0}\right)^2}} \tag{43}$$