MECA H 303:
Kinematics and dynamics of machines

Partim: Dynamics and vibrations

Exercise session 2

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1 P2.1

Let us consider the system represented in Fig. 1

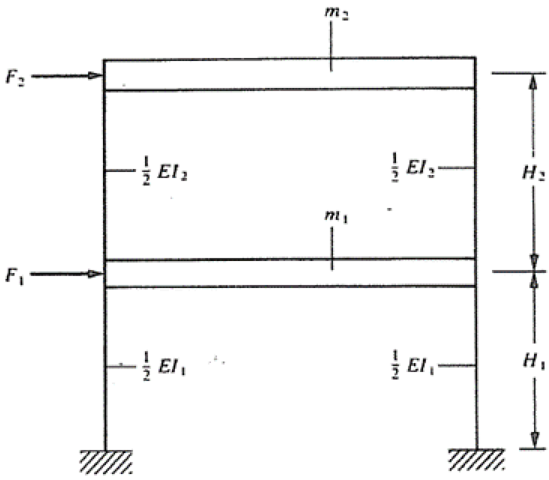

Figure 1: Simplified two-story building diagram

Assuming that the inertias (I_1, I_2), masses (m_1, m_2) and distances (H_1, H_2) are the same, the stiffness of each connection can be calculated with the equation

$$k = \frac{12EI}{H^3} \tag{1}$$

Assuming that there is only horizontal motion of the masses, the system in Fig. 1 can be represented by the mass-spring system shown in Fig. 2

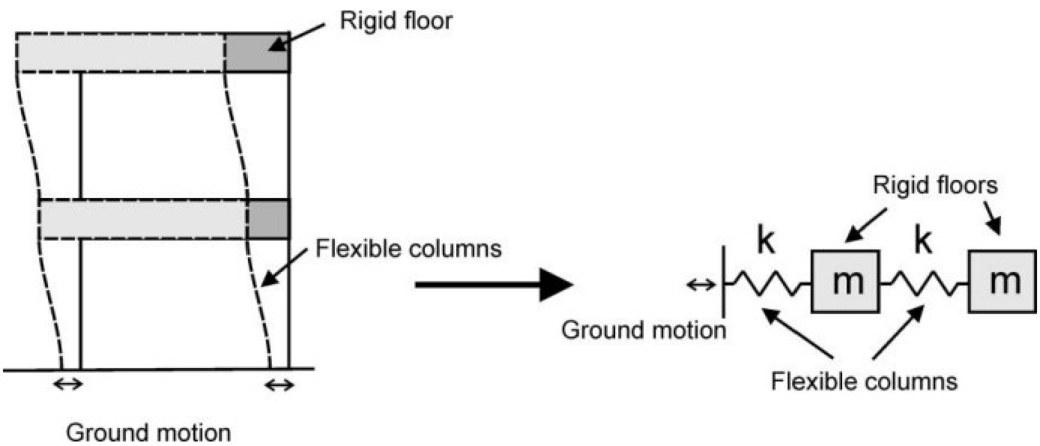

Figure 2: Mass-spring representation of a two story building subjected to horizontal motion

Equations of motion

The dynamic equations of motion can be extracted by applying Newton's law of motion to each mass, resulting in

$$\begin{aligned} m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = F_1 \\ m_2\ddot{x}_2 + k_2(x_2 - x_2) = F_2 \end{aligned} \tag{2}$$

Which can be expressed in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{3}$$

Natural frequencies

In order to obtain the natural frequencies of the system, let us first substitute the input and output variables assuming that they are harmonic in nature using $A = Ae^{i\omega t}$

$$-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \cancel{e^{i\omega t}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k2 & k_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \cancel{e^{i\omega t}} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \cancel{e^{i\omega t}} \tag{4}$$

From the general solution of the system (ie. $F_1 = F_2 = 0$) the characteristic equation can be written as

$$(K - \omega^2 M)X = 0 \tag{5}$$

Which admits a non-trivial solution if

$$det(K - \omega^2 M)X = 0 \tag{6}$$

Applying the above to eq. (4) results in

$$\begin{gathered} \det \left(\begin{bmatrix} k1 + k2 & -k2 \\ -k2k2 & \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right) = \begin{vmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{vmatrix} = 0 \\ m_1 m_2 \omega^4 - (k_1 m_2 + k_2 m_2 + k_2 m_1)\omega^2 + k_1 k_2 = 0 \end{gathered} \tag{7}$$

Calculating the determinant of the system and applying $k_1 = k_2, m_1 = m_2$ results in the following equation, who's roots can be obtained by solving for ω^2 thus obtaining the resonance frequencies of the system

$$\omega^4 m^2 - 3km\omega^2 + k^2 = 0 \tag{8}$$

$$\begin{gathered} \omega^2 = \frac{3km}{2m^2} \pm \frac{\sqrt{9k^2m^2 - 4m^2k^2}}{2m^2} = \frac{3km \pm km\sqrt{5}}{2m^2} \\ \omega^2 = \frac{k(3 \pm \sqrt{5}}{2m} \end{gathered} \tag{9}$$

2 P2.2

Let us consider the system in Fig. 3 which consists of two pendulums hanging from a rigid surface with equal masses on their free ends. At a vertical distance a from the fixed surface, a spring of stiffness k connects both pendulums.

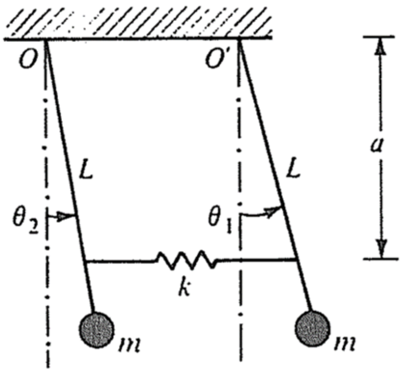

Figure 3: Connected double pendulum system

The initial conditions of the system are as follows

$$\begin{aligned} \theta_1(0) &= \theta_0 \\ \theta_2(0) = \dot{\theta}_1(0) &= \dot{\theta}_2(0) = 0 \end{aligned} \tag{10}$$

Due to the harmonic nature of the motion, the evolution of the variables θ_1, θ_2 as a function of time can be expressed in the form

$$\begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \end{Bmatrix} = [Z_{11}\cos(\omega_1 t) + Z_{12}\sin(\omega_1 t)] \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_1 + [Z_{21}\cos(\omega_2 t) + Z_{22}\sin(\omega_2 t)] \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_2 \tag{11}$$

Where Z_{ij} are constants depending on the initial conditions, ω_i are the natural frequencies of the system, and A_{ij} are the mode shapes of the system. All of which need to be obtained.

Equations of motion

In order to obtain the natural frequencies and associated mode shapes of the system, let's start with the equations of motion of the system by calculating the moments acting on each mass around their point of attachment to the fixed surface. We will begin with O'

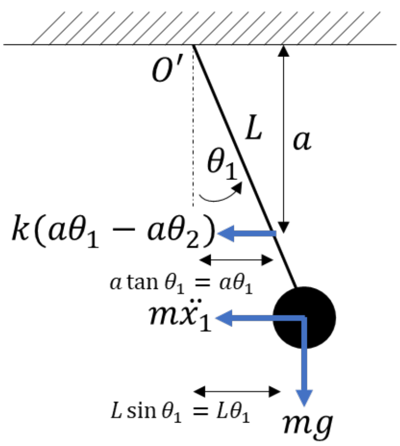

Figure 4: Moments about O'

The moments acting on the pendulum's pivot point are:

- The force of gravity acting vertically on the mass $F_g = -mg$ applied at a distance $L\sin\theta_1$ which, due to the approximation of small angles is simplified to $L\theta_1$
- The force related to the acceleration experienced by the mass $F_a = -m\ddot{x_1}$ applied at a distance L, where $x_1 = L\sin(\theta_1)$ which can be simplified to $x_1 = L\theta_1$ and consequently $\ddot{x_1} = L\ddot{|theta}_1$
- The restoring force of the spring which after applying the simplification of small angles and taking into account the contribution of both pendulums results in $F_k = -k(a\theta_1 - a\theta_2)$ applied at a distance a

The equation of motion for the first pendulum can now be written as

$$(mL\ddot{\theta_1})L + (mg)L\theta_1 + ka(\theta_1 - \theta_2)a = 0 \tag{12}$$

And analogously for the second pendulum

$$(mL\ddot{\theta_2})L + (mg)L\theta_2 + ka(\theta_2 - \theta_1)a = 0 \tag{13}$$

Written in matrix form

$$\begin{bmatrix} mL^2 & 0 \\ 0mL^2 & \end{bmatrix} \begin{Bmatrix} \ddot{\theta_1} \\ \ddot{\theta_2} \end{Bmatrix} + \begin{bmatrix} mgL + ka^2 & -ka^2 \\ -ka^2 & mgL + ka^2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14}$$

Natural frequencies and mode shapes

Let's calculate the natural frequencies of the system from the general solution $(K - \omega^2 M)\left\{A\right\} = 0$ by calculating $\det(K - \omega^2 M) = 0$

$$\begin{gathered} \det\left(\begin{bmatrix} mgL + ka^2 - \omega^2 mL^2 & -ka^2 \\ -ka^2 mgL + ka^2 - \omega^2 mL^2 & \end{bmatrix}\right) = 0 \\ (mgL + ka^2 - \omega^2 mL^2)^2 - (ka^2)^2 = 0 \\ mgL + ka^2 - \omega^2 mL^2 = \pm ka^2 \end{gathered} \tag{15}$$

Solving the quadratic equation for ω we find

$$\begin{gathered} \omega_1 = \pm\sqrt{\frac{g}{L}} \\ \omega_2 = \pm\sqrt{\frac{g}{L} + \frac{2ka^2}{mL^2}} = \pm\sqrt{\frac{mgL + ka^2}{mL^2}} \end{gathered} \tag{16}$$

Since we know that $mgL >> ka^2$, ω_2 can be simplified to

$$\omega_2 = \omega_1 = \pm\sqrt{\frac{g}{L}} \tag{17}$$

By applying the obtained resonance frequencies to the eigenvalue problem the mode shapes can be obtained, however by observation, since both resonance frequencies are of the same magnitude and opposite sign, the mode shapes will be equal in magnitude and of equal sign in one mode and opposite in the other, leading to the following results

$$\begin{gathered} A_{11} = 1 = A_{21} \\ A_{12} = 1 = -A_{22} \end{gathered} \tag{18}$$

Applying initial conditions

The mode shapes can be now applied to Eq. (11) by substituting $t = 0$ for the initial conditions of displacement, and differentiating the equation with respect to t and then applying $t = 0$ for the initial conditions of velocity. Doing this results in

$$\begin{gathered} Z_{11} = \frac{1}{2}\theta_0 = Z_{12} \\ Z_{21} = \frac{1}{2}\theta_0 = -Z_{12} \end{gathered} \tag{19}$$

Since the problem statement uses the variables ω_1, ω_2 in the solution, we will leave the resultant system of equations as a function of these variables

$$\begin{aligned} \theta_1(t) &= \frac{1}{2}\theta_0 \cos(\omega_1 t) + \frac{1}{2}\theta_0 \sin(\omega_2 t) \\ \theta_2(t) &= \frac{1}{2}\theta_0 \cos(\omega_1 t) - \frac{1}{2}\theta_0 \sin(\omega_2 t) \end{aligned} \tag{20}$$

All that is left to do is to rearrange both equations into the format stated in the exercise. In order to do that, the trigonometric relation in Eq. (21) can be applied

$$\begin{aligned} &\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \pm \sin\alpha \sin\beta \\ &\text{Where: } \alpha = \frac{\omega_2 - \omega_1}{2}t \ ; \ \beta = \frac{\omega_2 + \omega_1}{2}t \end{aligned} \tag{21}$$

Resulting in

$$\begin{aligned} \theta_1(t) &= \theta_0 \cos\left(\frac{\omega_2 - \omega_1}{2}t\right) \cos\left(\frac{\omega_2 + \omega_1}{2}t\right) \\ \theta_2(t) &= \theta_0 \sin\left(\frac{\omega_2 - \omega_1}{2}t\right) \sin\left(\frac{\omega_2 + \omega_1}{2}t\right) \end{aligned} \tag{22}$$

3 P2.3

A structure represented by a mass $m = 10\,kg$ and stiffness $k = 3\,kN/m$ is modified by adding a dynamic vibration absorber (DVA) of mass $m_a = 1\,kg$. The system is shown in Fig. 3

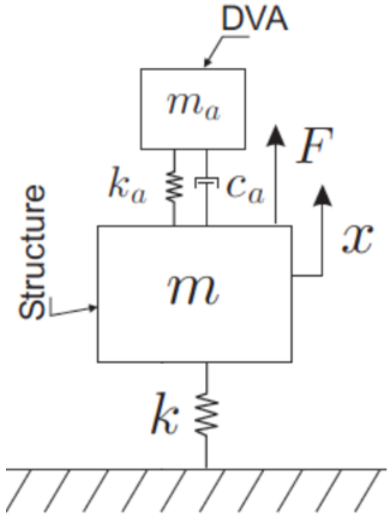

Figure 5: Structure with DVA

Since this is a damped DVA, the equal peak procedure introduced in the theory lectures can be applied. Independent of the damping in the DVA, the transfer function between a force applied to the main structure and the its displacement always passes by two points P, Q, whose vertical coordinates depend on the value of the natural frequency ratio $\nu = \frac{\omega_n}{\Omega}$ between the DVA (ω_n) and the main structure (Ω). In order to apply this methodology, two conditions must be fulfilled

1. The natural frequency ratio ν, must be chosen such that the verical coordinates of $P,\ Q$ are equal
2. The damping of the DVA ξ_a must be chosen such that $P,\ Q$ are maxima

The points $P,\ Q$ will be of the same height when the frequency ratio satisfies the following equation

$$\nu = \frac{\omega_n}{\Omega} = \frac{1}{1+\mu} \qquad \text{Where } \mu = \frac{m_a}{m} = 0.1 \tag{23}$$

This set of equations can be solved for ω_n and in turn obtain the stiffness k_a of the DVA

$$\omega_n = \Omega\frac{1}{1+\mu} = \sqrt{\frac{k}{m}}\left(\frac{1}{1+\mu}\right) = \sqrt{\frac{k_a}{m_a}} \qquad k_a = m_a\left(\Omega\frac{1}{1+\mu}\right)^2 = 248 \text{ N/m} \tag{24}$$

The optimal damping of the system is chosen such that $P,\ Q$ are maxima, which is done by the set of equations

$$\xi_{a,opt} = \sqrt{\frac{3\mu}{8(1+\mu)}} \qquad c_a = \xi_{a,opt}2m_a\omega_a = 5.82 \text{ N/(m/s)} \tag{25}$$

The maximum amplification will occur when there is no damping in the system, and is exclusively dependent on the mass ratio chosen for the system

$$H_{max} = (1+\mu)\sqrt{\frac{2+\mu}{\mu}} = 5.04 \tag{26}$$