MECA H 303:
Kinematics and dynamics of machines

Partim: Dynamics and vibrations

Exercise session 3

TA: Vicente Lafarga
Email: vlafarga@ulb.be
Phone: +32 2 650 67 47
Office: BEAMS UC2

P3.1

When looking at a Jeffcott rotor in the $X - Y$ plane, the resulting equations of motion are

$$\begin{aligned} m\ddot{x_c} + (c_n + c_r)\dot{x_c} + kx_c + c_r\Omega y_c = 0 \\ m\ddot{y_c} + (c_n + c_r)\dot{y_c} + kx_c - c_r\Omega x_c = 0 \end{aligned} \tag{1}$$

Note that these equations are equalled to zero because we are under free whirl conditions.
This system of equations can be transformed into complex coordinates for convenience in its manipulation by multiplying the second equations by i and adding it to the first equation satisfying

$$r_c = x_c + iy_c \tag{2}$$

Reducing Eq's (1) to

$$m\ddot{r_c} + (c_n + c_r)\dot{r_c} + (k - c_r\Omega)r_c = 0 \tag{3}$$

Let's now obtain the roots of this equation by applying the Laplace transform $r_c = r_o e^{st}$

$$ms^2\cancel{r_o} + (c_n + c_r)s\cancel{r_o} + (k - ic_r\Omega)\cancel{r_o} = 0 \tag{4}$$

Solving for s

$$s = -\frac{(c_n + c_r)}{2m} \pm \sqrt{\frac{(c_n + c_r)^2 - 4m(k - ic_r\Omega)}{4m^2}} \tag{5}$$

Let's separate the real and imaginary parts of the square root by using the following rule

$$\sqrt{a \pm ib} = \pm\sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} + i\,\mathrm{sgn}(b)\sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}} \tag{6}$$

Where

$$\begin{aligned} a = \frac{(c_n + c_r)^2 - 4mk}{4m^2} = \left(\frac{c_n + c_r}{2m}\right)^2 - \frac{k}{m} \\ b = \pm\frac{c_r\Omega}{m} \\ \Gamma = -a \end{aligned} \tag{7}$$

Resulting in

$$\begin{aligned} Re(s) = -\frac{c_n + c_r}{2} \pm \frac{1}{\sqrt{2}}\sqrt{\sqrt{\Gamma^2 + \left(\frac{\Omega c_r}{m}\right)^2} - \Gamma} \\ Im(s) = \pm\frac{\mathrm{sgn}(\Omega)}{\sqrt{2}}\sqrt{\sqrt{\Gamma^2 + \left(\frac{\Omega c_r}{m}\right)^2} + \Gamma} \end{aligned} \tag{8}$$

The system will be stable as long as the real part of the roots of the equations of motion remains negative. In order to verify this, let's substitute

$$\Omega = \sqrt{\frac{k}{m}}\left(1 + \frac{c_n}{c_r}\right) \tag{9}$$

into the real part of the equation of motion and check whether it results in a negative value (indicating stability), a positive value (indicating instability) or zero (indication the stability limit condition)

$$\Gamma^2 = \left(\frac{k}{m}\right)^2 - \frac{2k(c_n + c_r)^2}{4m^3} + \frac{(c_n + c_r)^4}{16}$$

$$\Omega^2 = \frac{k(c_n + c_r)^2}{mc_r^2}$$

$$\sigma_1 = -\frac{(c_n + c_r)}{2m} + \frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(\frac{k}{m}\right)^2 - \frac{k(c_n + c_r)^2}{2m^3} + \frac{(c_n + c_r)^4}{16m^4} + \frac{2k(c_n + c_r)^2}{m^3}} - \frac{k}{m} + \frac{(c_n + c_r)^2}{4m^2}}$$

Solving the square root inside of the square root (10)

$$\sqrt{\left(\frac{k}{m}\right)^2 - \frac{k(c_n + c_r)^2}{2m^3} + \frac{(c_n + c_r)^4}{16m^4} + \frac{2k(c_n + c_r)^2}{m^3}} = \frac{4km + (c_n + c_r)^2}{4m^2}$$

Reintroducing this into σ_1

$$\sigma_1 = -\frac{c_n + c_r}{2m} + \frac{1}{\sqrt{2}}\sqrt{\frac{4km + (c_n + c_r)^2}{4m^2} - \frac{k}{m} + \frac{(c_n + c_r)^2}{4m^2}}$$

$$\sigma_1 = -\frac{c_n + c_r}{2m} + \frac{1}{\sqrt{2}}\sqrt{\frac{(c_n + c_r)^2}{2m^2}} = -\frac{c_n + c_r}{2m} + \frac{c_n + c_r}{2m} = 0$$

Thus proving that the as long as Ω remains below $\sqrt{\frac{k}{m}}\left(1 + \frac{c_n}{c_r}\right)$ the system will be unconditionally stable. This stability condition is not to be confused with the critical speed of the system, which corresponds to the rotating speed equal to the natural frequency of the system. However, for the case of a Jeffcott rotor **without** damping, the stability condition does correspond to the critical speed of the system.

The maximum speed for which the system will be stable is therefore equal to the speed of the stability condition. One must therefore apply the numeric values given to the equation of the stability condition, resulting in

$$\Omega = \sqrt{\frac{k}{m}}\left(1 + \frac{c_n}{c_r}\right) = \sqrt{\frac{3000}{400}}\left(1 + \frac{4000}{200}\right) = 57.5\,\text{rad/s} = 549\,\text{rpm} \tag{11}$$

P3.2

Since the rotor is turning at a constant speed it means that the total energy injected by the motor must equal the total energy dissipated by the rotor

$$P_m = T_m\Omega = P_d \tag{12}$$

Where $\P_m$, T_m are the power and torque of the motor, and P_d is the power dissipated in the rotor. Let's take a look at the equation of motion of a Jeffcott rotor with damping. In this case, the motor is not free whirling, since it is being driven by a motor. Meaning that the sum of forces in the equation of motion is not equal to zero

$$m\ddot{r_c} + (c_n + c_r)\dot{r_c} + (k - ic_r\Omega)rc = m\epsilon\Omega^2 \exp^{i\Omega t} \tag{13}$$

Since the motion of the rotor is harmonic in nature we can substitute

$$r_c = r_0 \exp^{i\Omega t} \tag{14}$$

Resulting in

$$\begin{gathered}(-\Omega^2 m r_0 + c_n i\Omega r_0 + \cancel{c_r i\Omega r_0} + kr_0 - \cancel{c_r i\Omega r_0})\cancel{\exp^{i\Omega t}} = m\epsilon\Omega^2 \cancel{\exp^{i\Omega t}} \\ r_0(-\Omega^2 m + c_n i\Omega + k) = m\epsilon\Omega^2\end{gathered} \tag{15}$$

The only dissipative term in the above equation is that related to the non-rotating damping of the system, the dissipated power can be calculated as

$$P_d = -F_{cn}\dot{r_c} = c_n\Omega^2 \left|r_0\right|^2 \tag{16}$$

The maximum torque will be applied at the critical speed of the system, it is therefore easier to obtain $|r_0|^2$ as a function of the adimensional parameters Ω^* and ξ_n which relate the speed with the critical speed and the damping with the damping ratio through the following equations

$$\Omega = \Omega^* \sqrt{\frac{k}{m}} \tag{17}$$

$$c_n = 2\xi_n\sqrt{km} \tag{18}$$

Introducing eq's(17) and (18) into eq(15) we obtain

$$r_0 = \frac{\epsilon\Omega^{*2}}{\Omega^{*2})^2 + i\Omega^* 2\xi_n + 1} \tag{19}$$

Let's calculate the modulus of the expression

$$|r_0| = \frac{\epsilon\Omega^{*2}}{\sqrt{(1-\Omega^{*2})^2 + 4\Omega^{*2}\xi_n^2}} \tag{20}$$

$$|r_0|^2 = \frac{\epsilon^2\Omega^{*4}}{(1-\Omega^{*2})^2 + 4\Omega^{*2}\xi_n^2} \tag{21}$$

The required torque can now be calculated

$$T_m = \frac{P_d}{\Omega} = \frac{2k\xi_n\epsilon^2\Omega^{*5}}{(1-\Omega^{*2})^2 + 4\Omega^{*2}\xi_n^2} \tag{22}$$

In order to compute the maximum torque which will happen when rotating at the critical speed we substitute $\Omega^* = 1$ into the above equation resulting in

$$T_max = \frac{k\epsilon^2}{2\xi_n} \tag{23}$$

P3.3

A Jeffcott rotor with massless, compliant bearings can be represented by a two degree of freedom system where the first mass = 0, with the non-rotating damping and a stiffness representing the bearings connecting the first degree of freedom to ground, and the rotating damping and a stiffness representing the shaft connecting the bearings to the rotor. This can be seen in Fig. 1

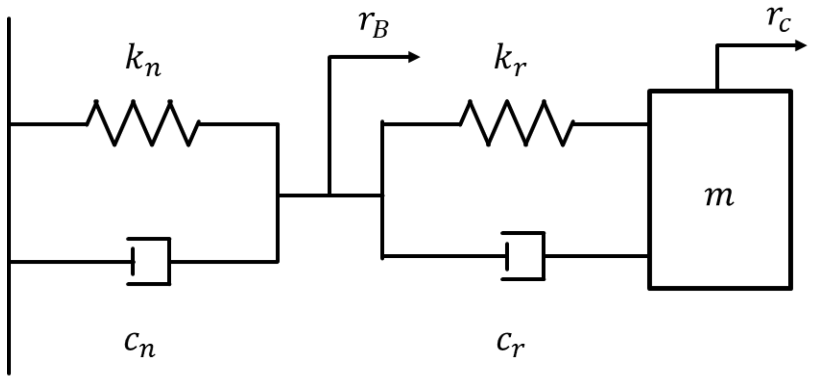

Figure 1: Simplified two-story building diagram

The equations of motion for the bearings and the rotor shaft are respectively

$$c_n\dot{r_B} + c_r(\dot{r_B} - \dot{r_c}) + k_n r_B + k_r(r_B - r_c) + i\Omega c_r(r_B - r_c) = 0 \tag{24}$$

$$m\ddot{r_c} + c_r(\dot{r_c} - \dot{r_B}) + k_r(r_c - r_B) + i\Omega c_r(r_c - r_B) = m\epsilon\Omega^2 \exp i\Omega t \tag{25}$$

Which can be written in matrix form as

$$\begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{r_B} \\ \ddot{r_c} \end{Bmatrix} + \begin{bmatrix} c_n + c_r & -c_r \\ -c_r & c_r \end{bmatrix} \begin{Bmatrix} \dot{r_B} & \dot{r_c} \end{Bmatrix} + \left(\begin{bmatrix} k_n + k_r & -k_r \\ -k_r & k_r \end{bmatrix} + i\Omega c_r \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \begin{Bmatrix} r_B \\ r_c \end{Bmatrix} = \begin{Bmatrix} 0 \\ m\epsilon\Omega^2 \end{Bmatrix} \exp i\Omega t \tag{26}$$

The resulting system of equations of the undamped Jeffcott rotor is

$$\begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{r_B} \\ \ddot{r_c} \end{Bmatrix} + \begin{bmatrix} k_n + k_r & -k_r \\ -k_r & k_r \end{bmatrix} \begin{Bmatrix} r_B \\ r_c \end{Bmatrix} = \begin{Bmatrix} 0 \\ m\epsilon\Omega^2 \end{Bmatrix} \exp i\Omega t \tag{27}$$

Since the critical speed is equal to the resonance frequency of the system, we can now proceed as was done in the previous exercise session for two degree of freedom systems by writing the characteristic equation $\det(K - \Omega^2 M) = 0$ and solving for Ω which results in

$$\Omega = \sqrt{\frac{k_n k_r}{(k_n + k_r)m}} \tag{28}$$

Which is both the resonance frequency and the critical speed of the system