# RESPONSE SPECTRA

Arnaud Deraemaeker
Arnaud.Deraemaeker@ulb.be

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# TYPES OF RESPONSE SPECTRA

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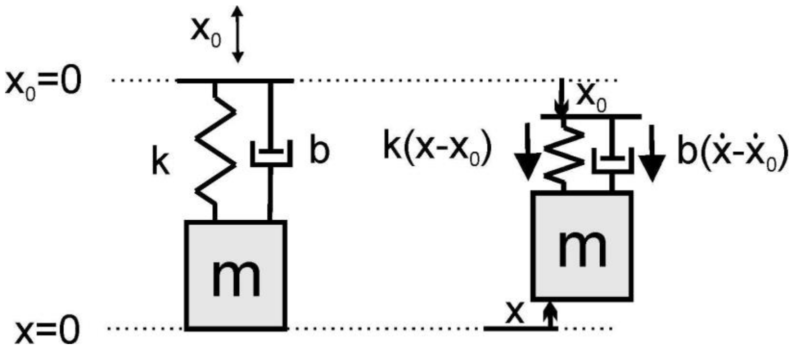
## Equation of motion




$$m\ddot{x} = -k(x - x_0) - b(\dot{x} - \dot{x}_0)$$

$$x_r = x - x_0$$

$$m\ddot{x}_r + b\dot{x}_r + kx_r = -m\ddot{x}_0$$

Excitation

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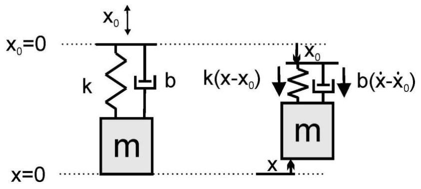
## Response to ground acceleration in the time domain




$$m\ddot{x} = -k(x - x_0) - b(\dot{x} - \dot{x}_0)$$
$$x_r = x - x_0$$
$$m\ddot{x}_r + b\dot{x}_r + kx_r = -m\ddot{x}_0$$

Earthquake base acceleration

Duhamel's integral

$$x(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = f(t) * h(t)$$

Impulse response

$$h(t) = \frac{e^{-\xi\omega_n t}}{m\omega_d}\sin(\omega_d t)$$

$$\longrightarrow \quad x_r(t) = \int_{-\infty}^{\infty} -\ddot{x}_0(\tau)\frac{e^{-\xi\omega_n(t-\tau)}}{\omega_d}\sin(\omega_d(t-\tau))d\tau$$

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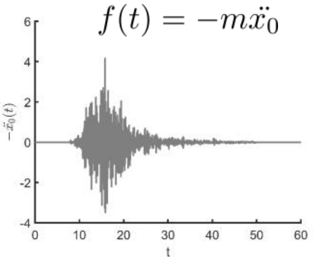
## Santa Cruz earthquake (1990)

$$f(t) = -m\ddot{x}_0$$

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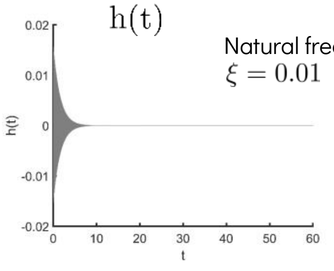
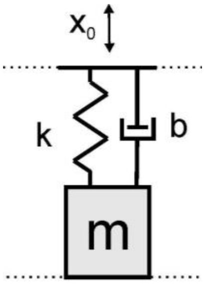
h(t)




Natural frequency = 6 Hz
$\xi = 0.01$

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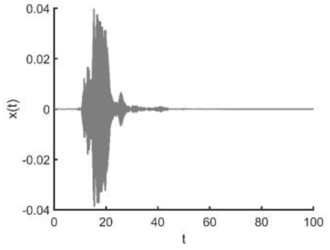
Response to earthquake excitation

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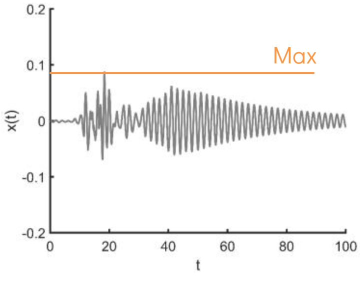
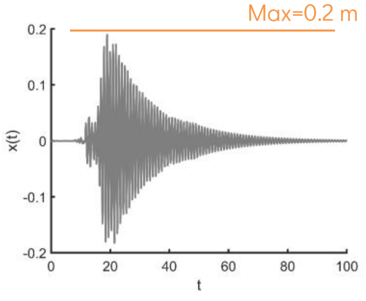
## Response spectra

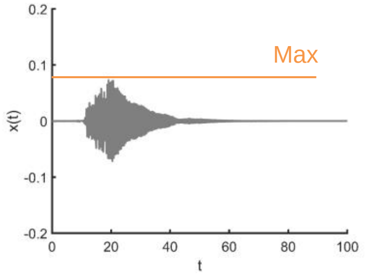
$$x_r(t) = \int_{-\infty}^{\infty} -\ddot{x}_0(\tau)\frac{e^{-\xi\omega_n(t-\tau)}}{\omega_d}\sin(\omega_d(t-\tau))d\tau$$




Natural frequency = 0.5 Hz
$\xi = 0.01$




Natural frequency = 1 Hz
$\xi = 0.01$




Natural frequency = 2 Hz
$\xi = 0.01$

- One value in a response spectrum corresponds to the maximum of the absolute value of the amplitude of a one dof system with given natural frequency and damping coefficent.
- It is built by varying the natural frequency of the one dof system, and taking the associated period T=1/f for the x-axis

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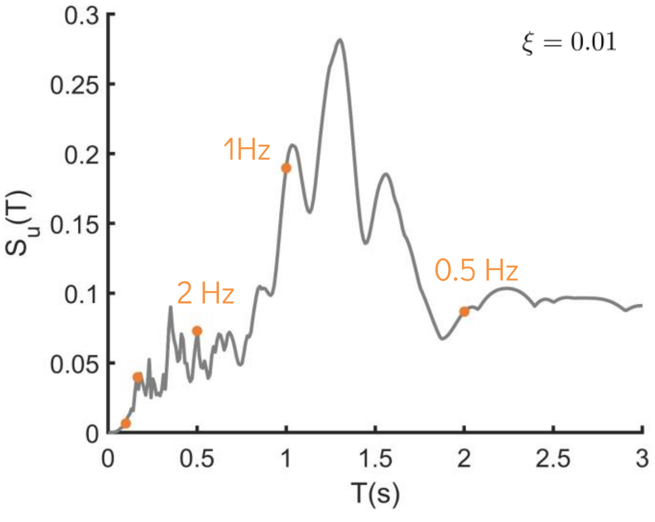
## Displacement response spectrum

$$S_u(\xi, \omega_n) = |x_r(t)|_{max}$$

Santa Cruz Earthquake




- The x-axis does not correspond to the frequency of excitation
- It corresponds to the eigenperiod of the one dof system considered
- The natural frequency of the corresponding system is f=1/T
- The response spectrum is specific to an earthquake

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## Response spectra

$$x_r(t) = \int_{-\infty}^{\infty} -\ddot{x}_0(\tau)\frac{e^{-\xi\omega_n(t-\tau)}}{\omega_d}\sin(\omega_d(t-\tau))d\tau$$

Relative displacement spectrum
$S_u(\xi, \omega_n) = |x_r(t)|_{max}$

Relative velocity spectrum
$S_v(\xi, \omega_n) = |\dot{x}_r(t)|_{max}$

Absolute acceleration spectrum
$S_a(\xi, \omega_n) = |\ddot{x}_0(t) + \ddot{x}_r(t)|_{max}$

Pseudo velocity response spectrum $S_{pv}(\xi, \omega_n) = \omega_n S_u(\xi, \omega_n)$

Pseudo acceleration response spectrum $S_{pa}(\xi, \omega_n) = \omega_n^2 S_u(\xi, \omega_n)$

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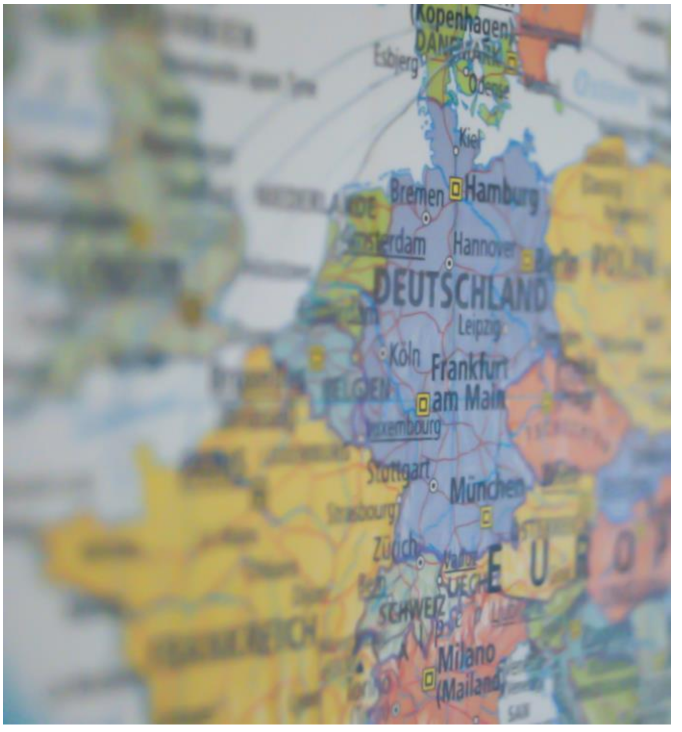
# RESPONSE SPECTRA IN THE EUROCODE 8

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## Response spectra in the Eurocode

For an undamped one dof system :

$$m\ddot{x}_r + kx_r = -m\ddot{x}_0$$

$$\ddot{x}_r + \omega_n^2 x_r = -\ddot{x}_0$$

$$\ddot{x}_r + \ddot{x}_0 = -\omega_n^2 x_r$$

$$S_{pa} = S_e = \omega_n^2 S_u \simeq S_a$$

→ The response spectrum specified in the Eurocode ($S_e$) is an approximation of the total acceleration, valid when damping is small

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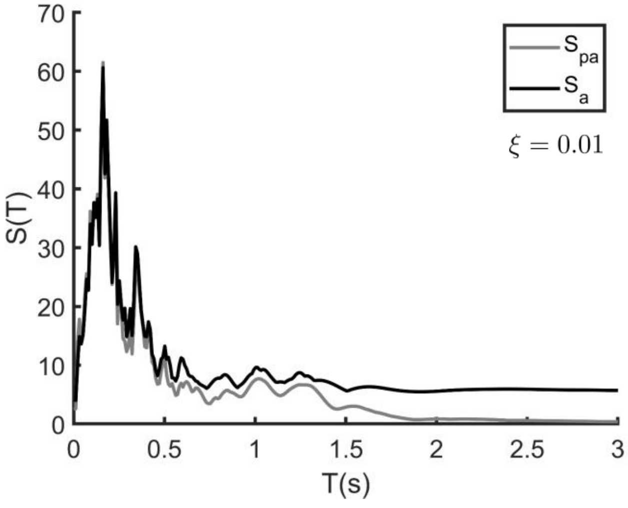
## Response spectra




Santa Cruz Earthquake

$$S_a(\xi, \omega_n) = |\ddot{x}_0(t) + \ddot{x}_r(t)|_{max}$$
$$S_{pa}(\xi, \omega_n) = \omega_n^2 S_u(\xi, \omega_n)$$

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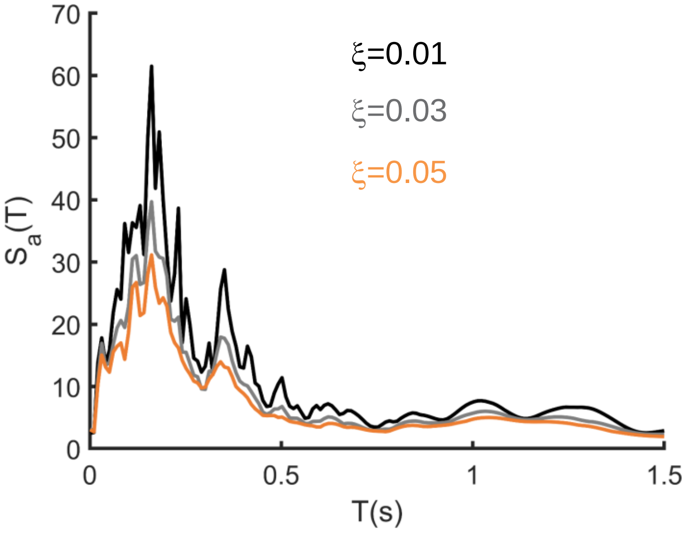
## Response spectra : influence of damping




Santa Cruz Earthquake

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# Response spectra in the Eurocode

Horizontal elastic spectrum

$$0 \le T \le T_B : S_e(T) = a_g S \left(1 + \frac{T}{T_B}(2.5\eta - 1)\right)$$

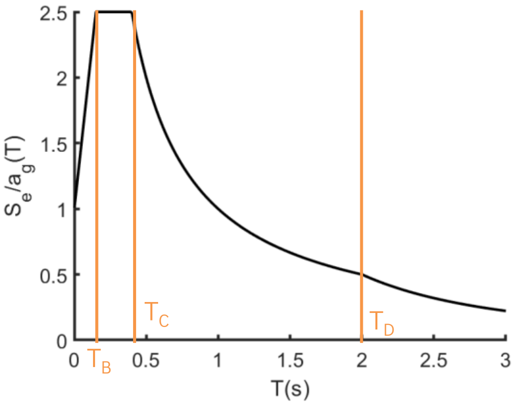
$$T_B \le T \le T_C : S_e(T) = a_g S\ 2.5\eta$$

$$T_C \le T \le T_D : S_e(T) = a_g S 2.5\eta \frac{T_C}{T}$$

$$T_D \le T \le 4s : S_e(T) = a_g S 2.5\eta \frac{T_C T_D}{T^2}$$

$a_g$= ground acceleration
S=soil factor
$\eta$=damping correction factor (=1 for 5% damping)

$$S_e(\xi, \omega_n) = \omega_n^2 S_u(\xi, \omega_n)$$

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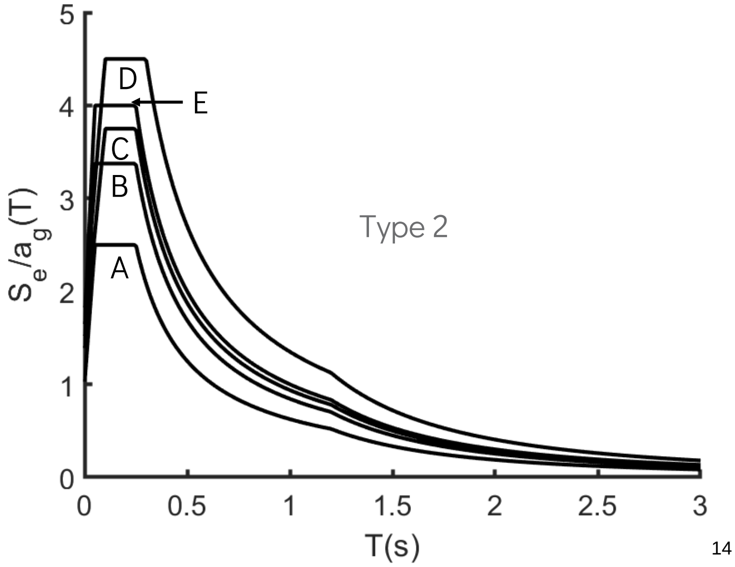
# Response spectra in the Eurocode

| Ground type | $S$ | $T_B$ | $T_C$ |
|---|---|---|---|
| A | 1.0 | 0.15 | 0.4 |
| B | 1.2 | 0.15 | 0.5 |
| C | 1.15 | 0.2 | 0.6 |
| D | 1.35 | 0.2 | 0.8 |
| E | 1.4 | 0.15 | 0.5 |

Type 1

| Ground type | $S$ | $T_B$ | $T_C$ |
|---|---|---|---|
| A | 1.0 | 0.05 | 0.25 |
| B | 1.35 | 0.05 | 0.25 |
| C | 1.5 | 0.1 | 0.25 |
| D | 1.8 | 0.1 | 0.3 |
| E | 1.6 | 0.05 | 0.25 |

Type 2 (Belgium)

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## Design spectrum vs elastic spectrum

The design spectrum takes into account the inelastic behavior using the behavior factor $q$, and adds a lower bound factor β

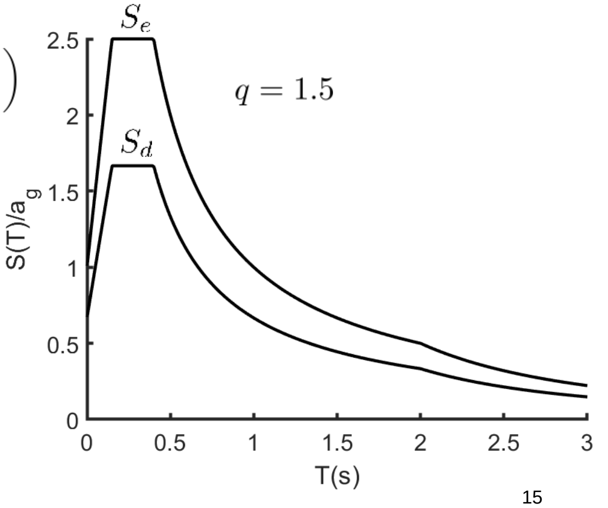
$$0 \leq T \leq T_B : S_d(T) = a_g S \left(\frac{2}{3} + \frac{T}{T_B}\left(\frac{2.5}{q} - \frac{2}{3}\right)\right)$$

$$T_B \leq T \leq T_C : S_d(T) = a_g S\, \frac{2.5}{q}$$

$$T_C \leq T \leq T_D : S_d(T) \begin{cases} = a_g S \frac{2.5}{q} \frac{T_C}{T} \\ \geq \beta a_g \end{cases}$$

$$T_D \leq T : S_d(T) \begin{cases} = a_g S \frac{2.5}{q} \frac{T_C T_D}{T^2} \\ \geq \beta a_g \end{cases}$$

$S_d$ is the design spectrum

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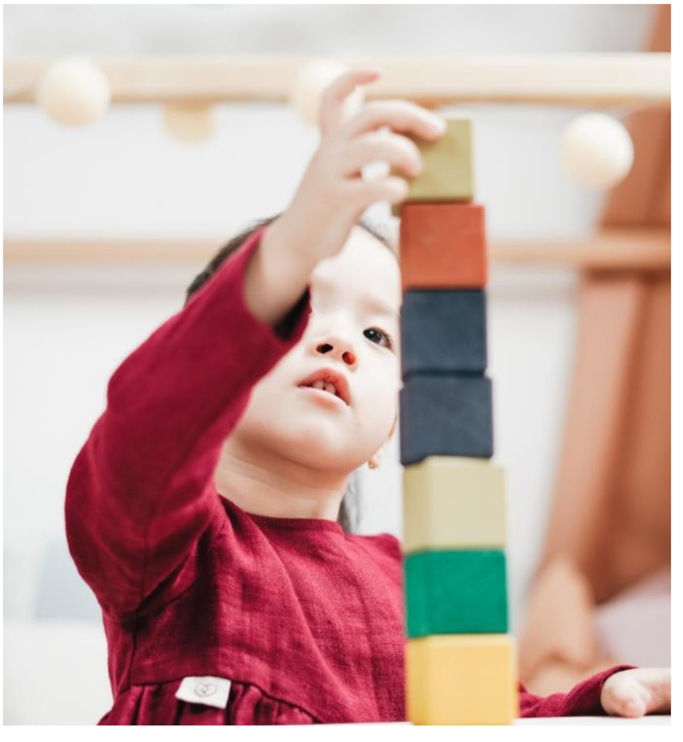
## RESPONSE SPECTRA FOR MDOF SYSTEMS



## Response spectra for MDOF systems

$$\ddot{z_{ri}} + 2\xi_i\omega_i\dot{z_{ri}} + \omega_i^2 z_{ri} = -\frac{\Gamma_i}{\mu_i}\ddot{x}_0 \qquad \Gamma_i = \psi_i^T M T$$


$$z_{i,max} = |z_i(t)|_{max} = \frac{\Gamma_i}{\mu_i} S_u(\xi_i, \omega_i)$$

$$x_{i,max} = z_{i,max}\ \psi_i$$ Contribution of mode i to the displacement vector

$$\ddot{x}_{a,i,max} = \frac{\Gamma_i}{\mu_i} S_{pa}(\xi_i, \omega_i)\ \psi_i$$

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## Response spectra for MDOF systems

Modal combinations

**Absolute sum method (ABSSUM)**

$$r_{max} = \sum_{i=1}^{n} |r_i|$$

→ Leads to over estimation

**Square root of sum of squares (SRSS)**

$$r_{max} = \sqrt{\sum_{i=1}^{n} r_i^2}$$

Most often used in practice

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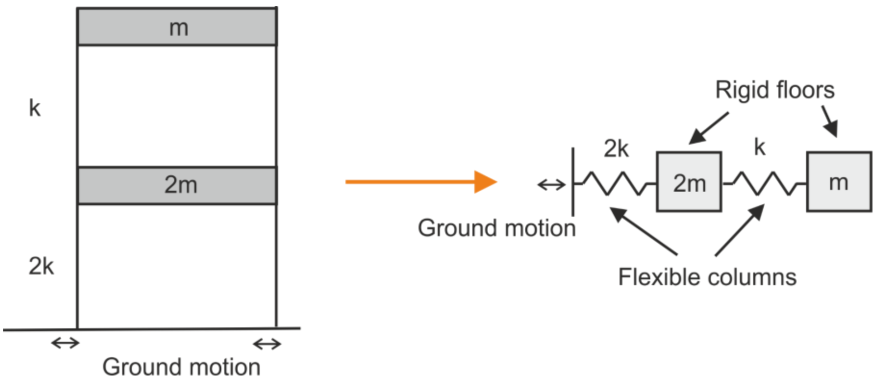
## Example of a two-floor building




$$k = 197\ 392\ N/m,\ m = 2500kg,\ \xi = 0.05$$

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## Example of a two-floor building : application of Eurocodes

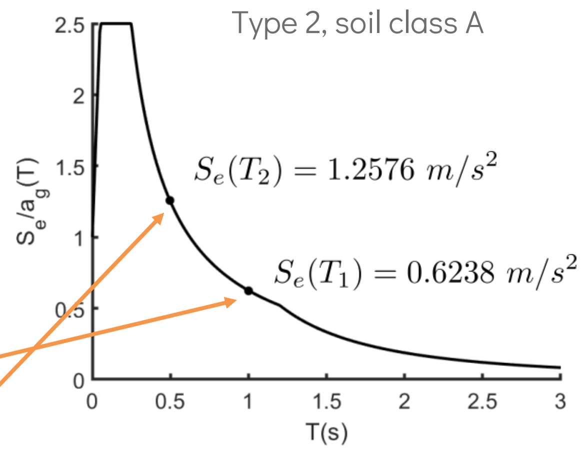
$$K = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} \qquad M = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}$$

Mode shapes and natural frequencies

$$\Psi = \begin{bmatrix} 0.5 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\omega_1 = 2\pi\ rad/s \qquad T_1 = 1s$$

$$\omega_2 = 4\pi\ rad/s \qquad T_2 = 0.5s$$

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# Example of a two-floor building

Design spectrum

$$S_e(T_1) = 0.6238 \ m/s^2 \qquad S_d(T_1) = 0.4158 \ m/s^2$$

$$S_e(T_2) = 1.2576 \ m/s^2 \quad \xrightarrow{q = 1.5} \quad S_d(T_2) = 0.8384 \ m/s^2$$

Modal participation factors

$$\frac{\Gamma_1}{\mu_1} = \frac{\psi_1^T M \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\psi_1^T M \psi_1} = \frac{4}{3} \qquad \frac{\Gamma_2}{\mu_2} = \frac{\psi_2^T M \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\psi_2^T M \psi_2} = -\frac{1}{3}$$

Max modal displacements $a_g = 1m/s^2$

$$Z_{1,max} = \frac{\Gamma_1}{\mu_1} a_g \ S_d(T_1)/\omega_1^2 = 0.0140m$$

$$Z_{2,max} = \frac{\Gamma_2}{\mu_2} a_g \ S_d(T_2)/\omega_2^2 = -0.0018m$$

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# Example of a two-floor building

Max displacement of top of building :

$$X_{1,max} = Z_{1,max}\psi_1 = 0.014 \ m \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$X_{2,max} = Z_{2,max}\psi_2 = -0.0018 \ m \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Top floor displacement, contribution of each mode

$$X_{SRSS,TOP} = \sqrt{X_{1,max}(2)^2 + X_{2,max}(2)^2} = 0.0142 \ m$$

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## Example of a two-floor building

Shear force at the bottom of ground level columns

$$F_{1,max} = 2kZ_{1,max}\psi_1(1) = 2.77\ kN$$

$$F_{2,max} = 2kZ_{2,max}\psi_2(1) = 682N$$

Shear force ground level columns, contribution of each mode

$$F_{SRSS,BOT\_SHEAR} = \sqrt{F_{1,max}(2)^2 + F_{2,max}(2)^2} = 2.859kN$$



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