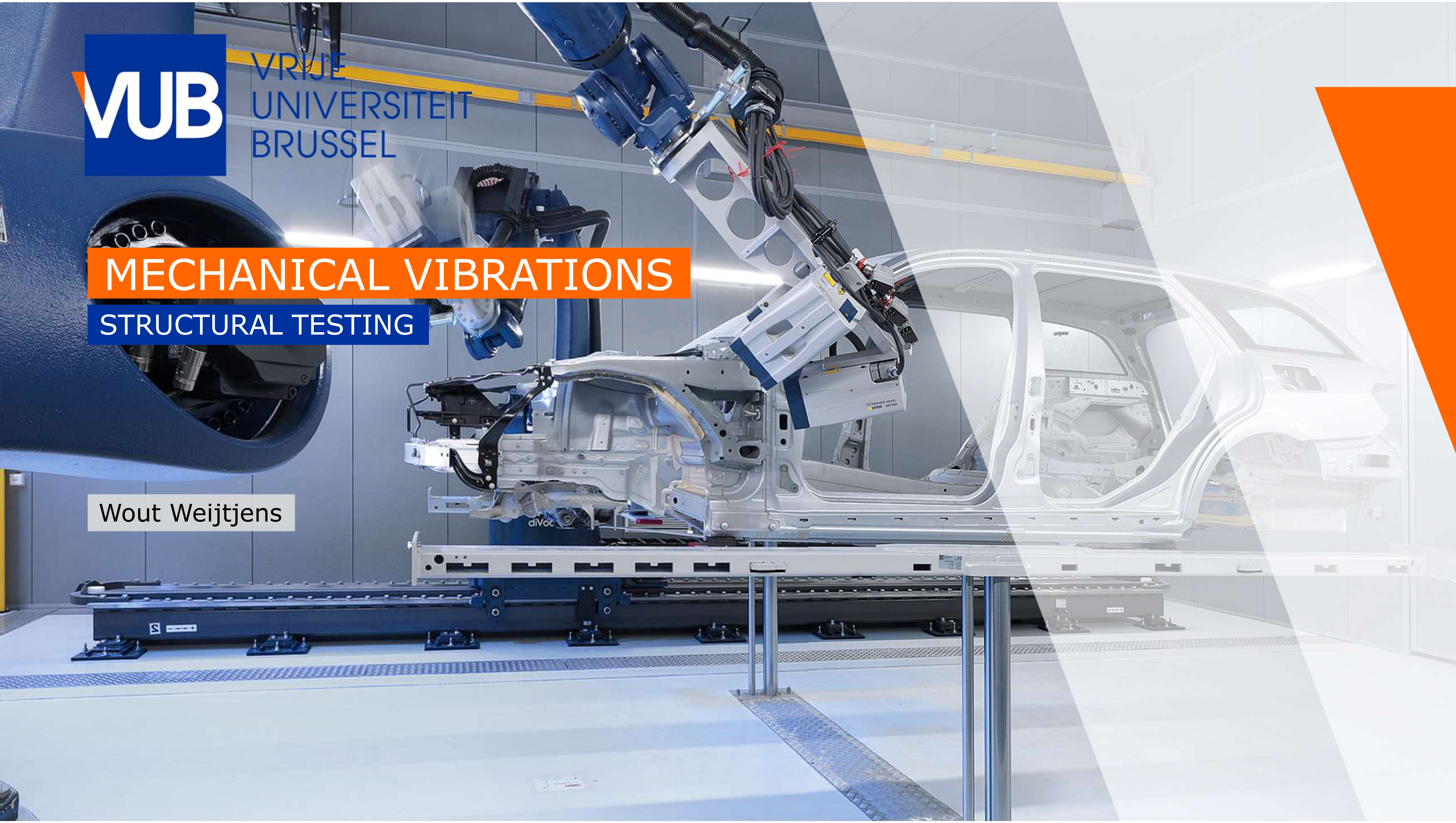


MECHANICAL VIBRATIONS

STRUCTURAL TESTING

Wout Weijtjens

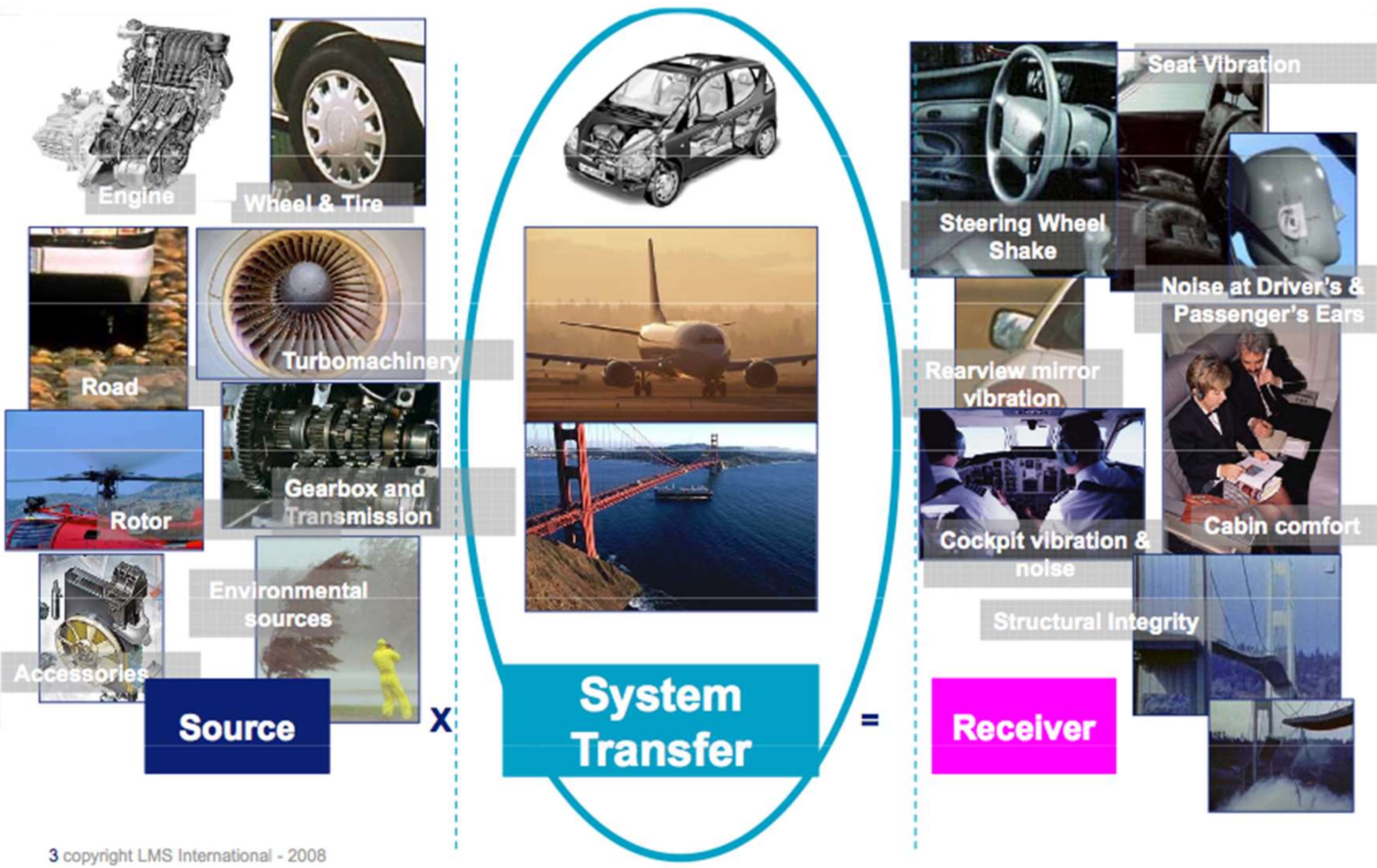
BASIC MODEL

SOURCE X SYSTEM TRANSFER = RECEIVER

Any noise or vibration perceived by the receiver is the product of a Source passing a System.

Want to increase passenger comfort?

Either you remove the source, or you optimise the transfer : **System analysis**



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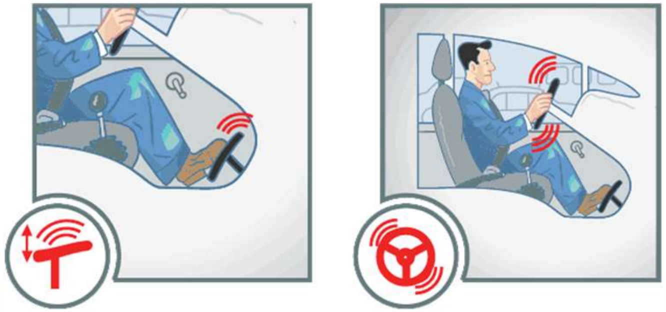
BRAKE THICKNESS VARIATION INDUCED VIBRATIONS (BTV)

AN EXAMPLE

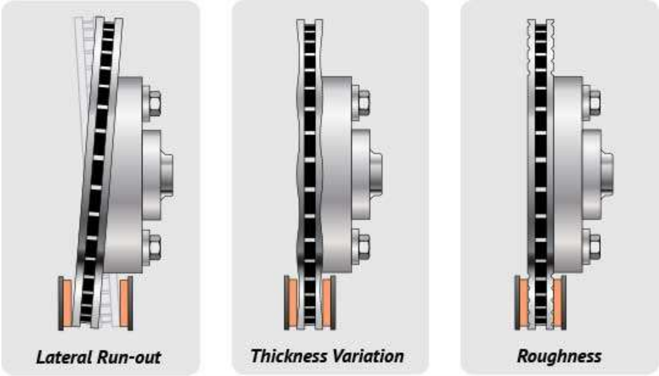


Vibrations induced in the steering wheel (small rotations) when applying the brake: a.k.a. Brake judder.

- *when I brake around X km/h my steering wheel starts to shake*

Caused by a deformation in the brake disk resulting in a non-constant force applied by the brake.

The frequency of the force variation is directly related to the rotational speed of the brake disk and the order of the deformation.



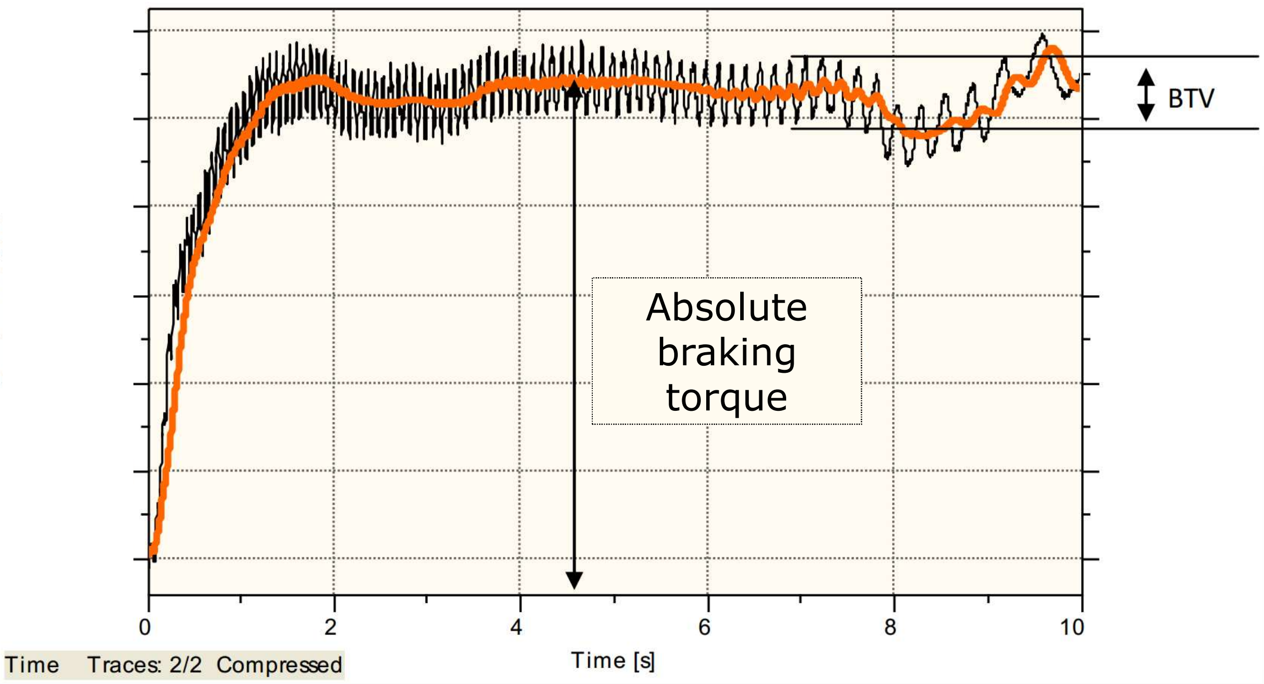
BRAKE THICKNESS VARIATION INDUCED VIBRATIONS

BEHAVIOR

Example of brake torque variations during a braking event.

One can nicely see that the BTV frequency decreases with time as the car decelerates

However driver will only notice this at a very specific speed : e.g. 140km/h.

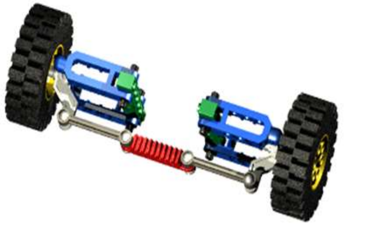


BRAKE THICKNESS VARIATION INDUCED VIBRATIONS

PERCEPTION

So why does the driver only notice it a given speed?

The BTV themselves are not really noticeable to the average driver, but when the BTV frequency matches the first resonance frequency of the steering rig.



Vibrations are passed through the steering rig into the steering wheel!

BRAKE THICKNESS VARIATION INDUCED VIBRATIONS

MODEL

Brake thickness variations X Steering rig dynamics = Steering wheel vibrations

BRAKE THICKNESS VARIATION INDUCED VIBRATIONS

SOLUTIONS

BRAKE THICKNESS VARIATION INDUCED VIBRATIONS

SOLUTIONS

Reduce source : avoid Brake thickness variations (but sometimes unavoidable, or too expensive to avoid)

Clever design transfer path : reduce discomfort to driver by designing the first steering wheel resonance at a frequency outside the normal driving window (e.g. above 140 km/h)

Remove transfer path : remove physical connection between steering wheel and steering rig, e.g. drive-by-wire

WITH THE RETURN OF THE ELECTRIC CAR

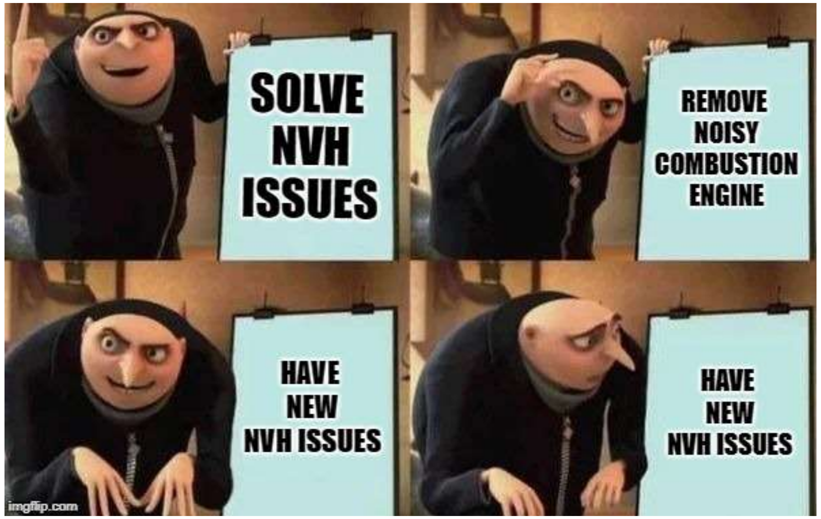
IS NVH A THING OF THE PAST?

Yes, electric cars don't have the noisy motor….

To <u>drown out </u>all other noise

Efforts to extend range, by reducing weight:

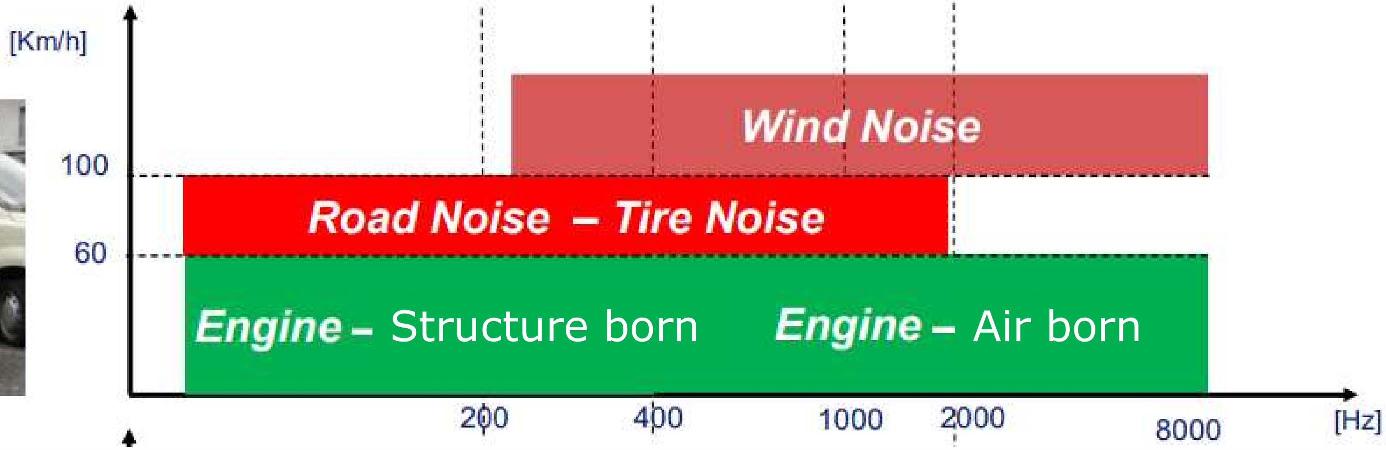
- Preferably less isolation
- New materials



WITH THE DAWN OF THE ELECTRIC CAR

IS NVH A THING OF THE PAST?

ICE

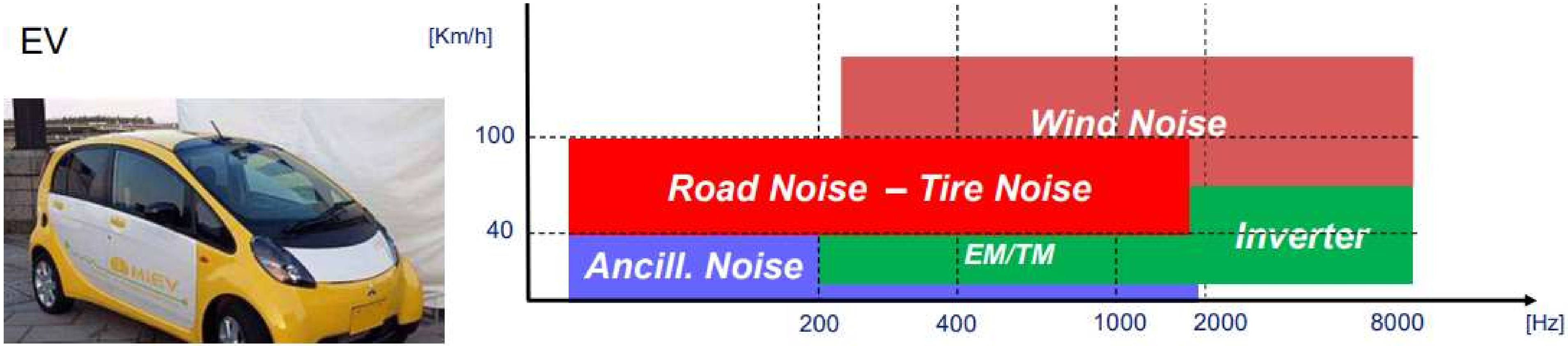


ICE : Internal Combustion Engine

Source : Vibro-acoustic engineering challenges in (hybrid and) electric vehicles, Siemens PLM

WITH THE DAWN OF THE ELECTRIC CAR

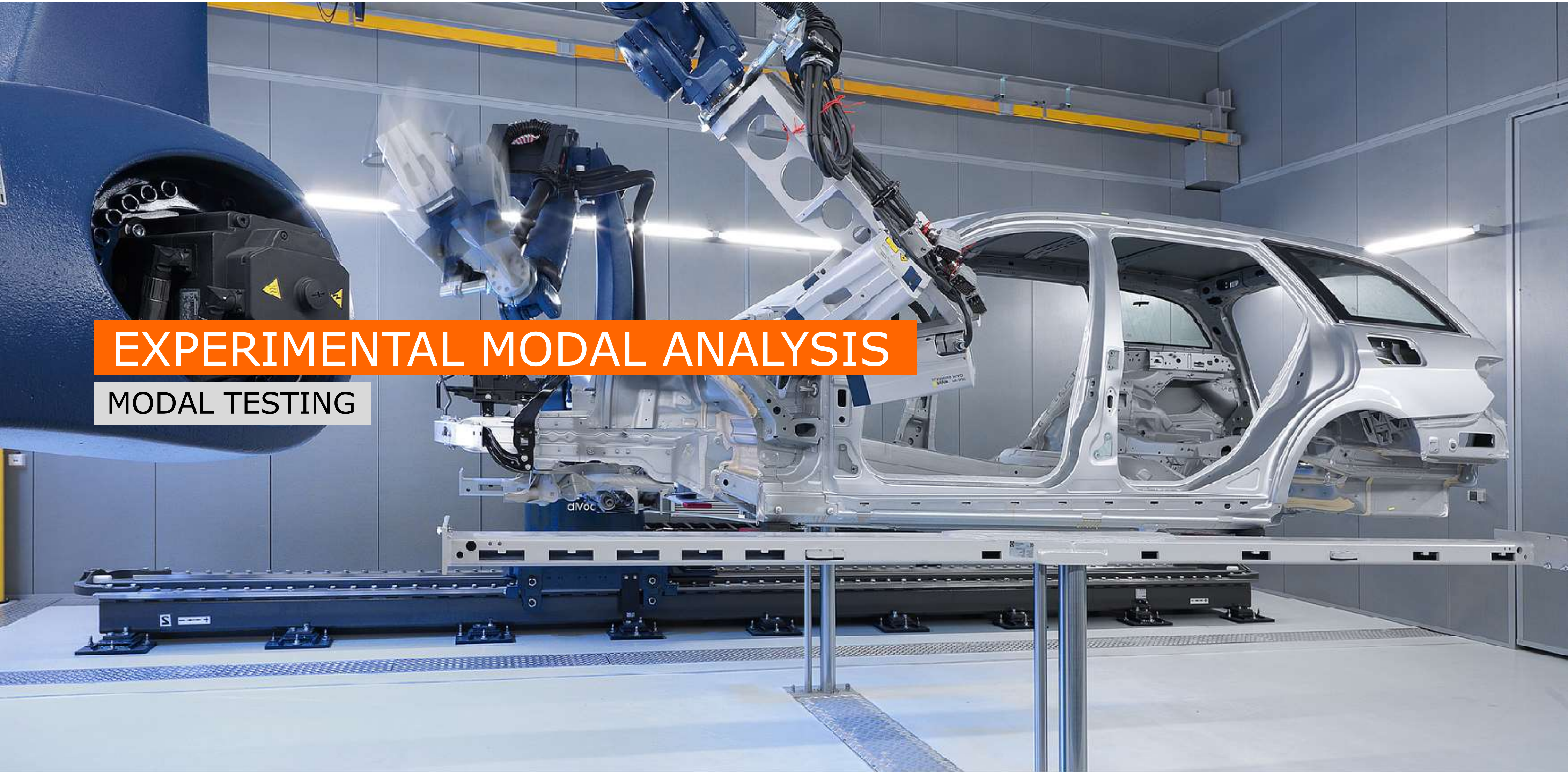
IS NVH A THING OF THE PAST?



- Road noise - Tire Noise plays a bigger role (at earlier speeds)
- New sources of noise *appear*
  - Ancillary noise, which was never an issue before : e.g. Air conditioning, Power steering motor
  - Electrical components : Electric Motor (EM), Invertor, Battery cooling
  - Complex gearings in Hybrid EV
- New types of noise
  - Tonal : Constant sharp tone, perceived as very annoying
  - Controlled e-Sound : Sound generated to alert pedestrians

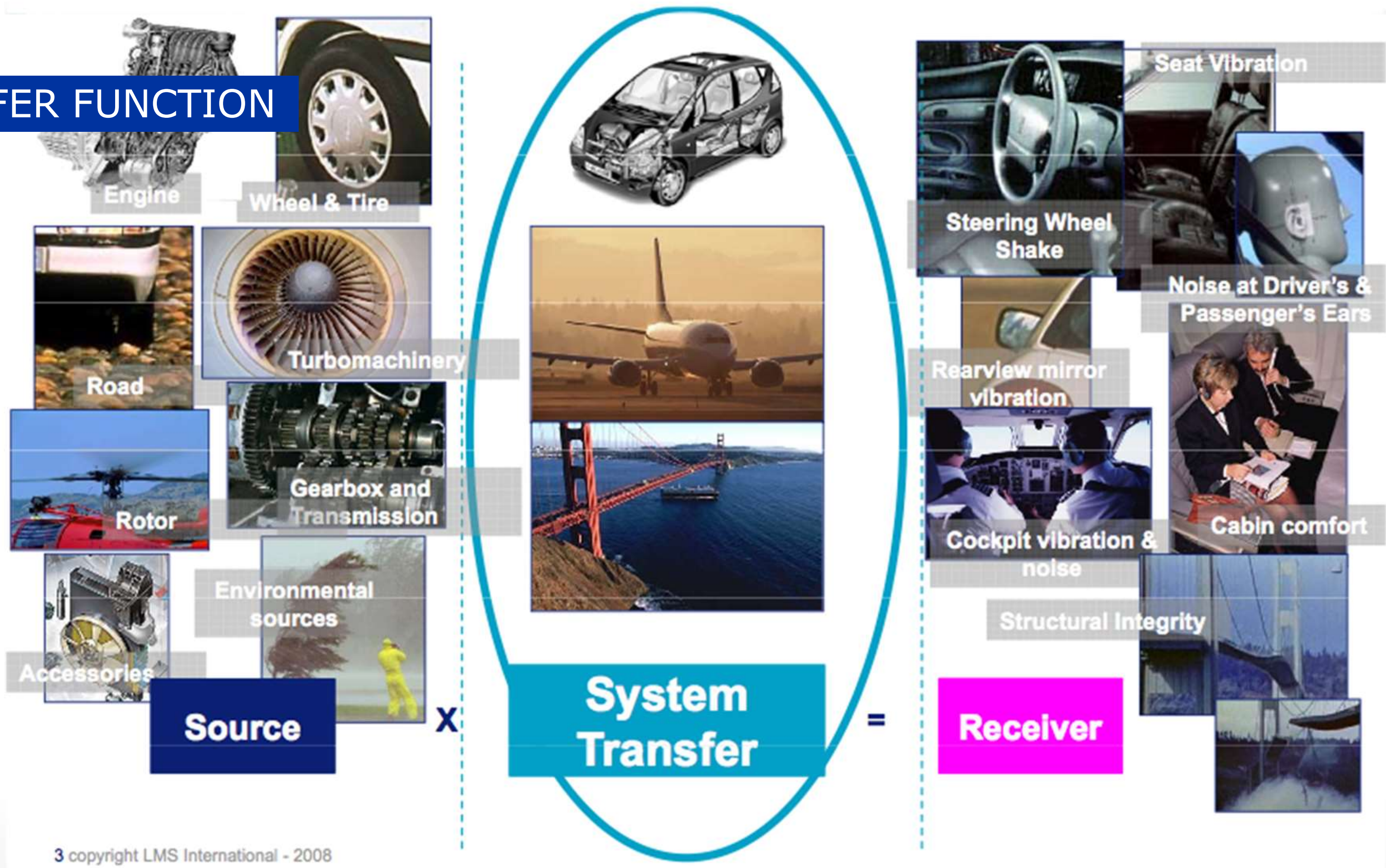
EXPERIMENTAL MODAL ANALYSIS

MODAL TESTING



MODAL TESTING

ESTIMATING THE TRANSFER FUNCTION



MODAL TESTING

ESTIMATING THE TRANSFER FUNCTION

Known force

$\vec{F}$

X

Transfer function

H

The big unknown

=

Recorded response

$\vec{X}$

MODAL TESTING

ESTIMATING THE TRANSFER FUNCTION

Known force		Transfer function		Recorded response
F	X	**H**	=	**X**
		The big unknown		

MODAL TESTING

STRUCTURE

- Strategies to Modal testing
  - Input measurements (F)
  - Output measurements (X)
- Designing a modal testing setup
- Signal processing
  - Process your measurements
  - Choosing an input signal
  - Compute the transfer function
- Data evaluate your data

-> Next topic : use this data for modal parameter estimation

STRATEGIES TO MODAL TESTING

GETTING THE MODAL PARAMETERS FROM MEASUREMENTS

- Known input : Shaker or impulse hammer
- Known output : Accelerometers or Laser Doppler Vibrometer

Outcome :

Frequency response function (FRF)

STRATEGIES TO MODAL TESTING

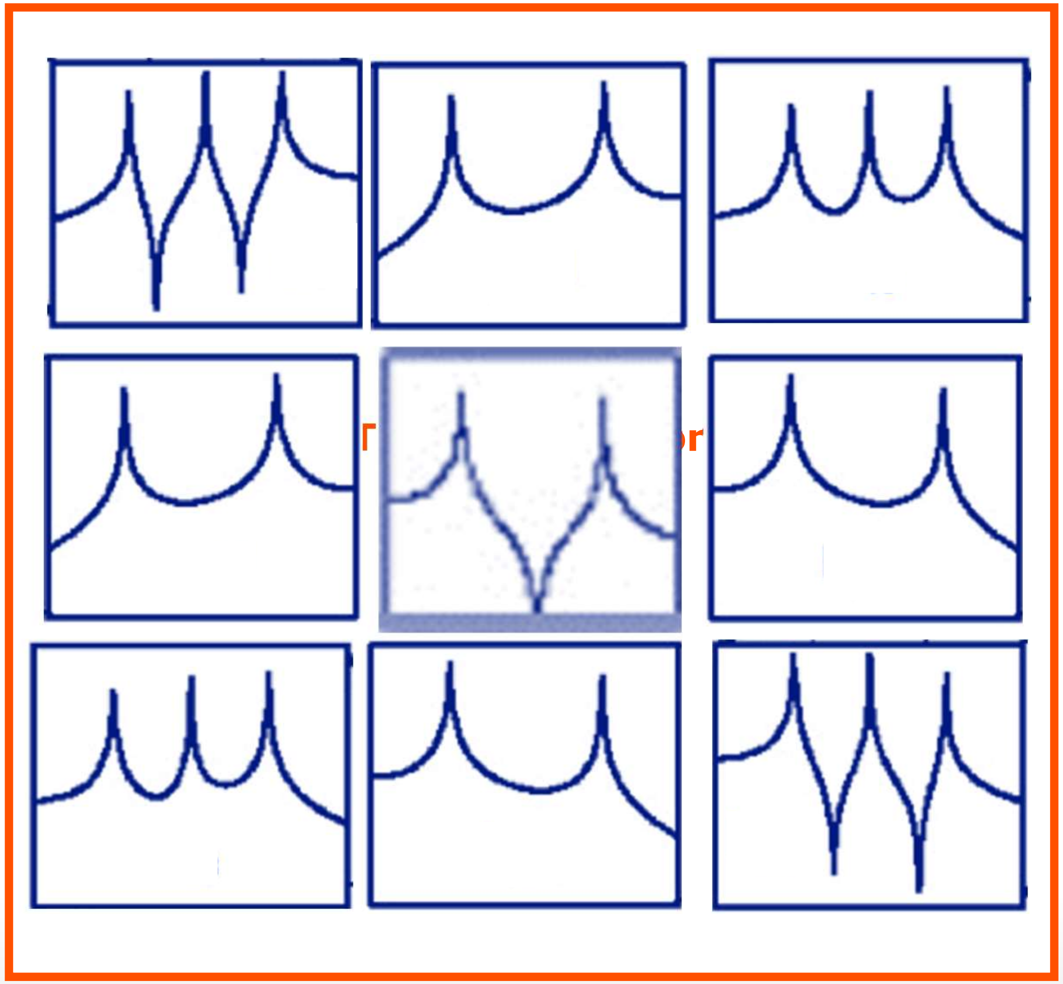
TRANSFER FUNCTION

The transfer function is 2 Dimensional. For an N-DOF system its dimensions are N x N.

i.e. there is a transfer function between each input and each output

But we don't have to measure all N^2 transfer functions because of the **reciprocity of the system,** i.e. the matrix is symmetric

It is sufficient to measure a single row or column.



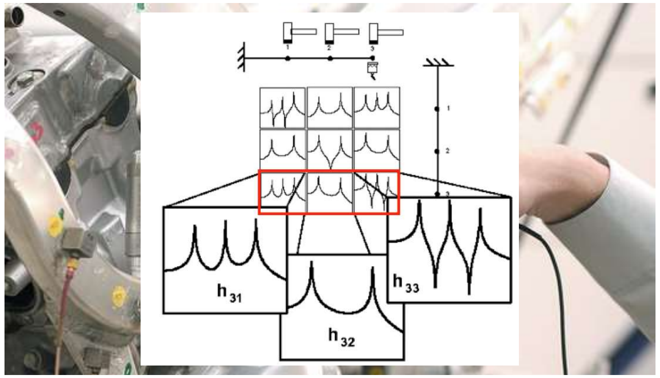
STRATEGY
ROVING INPUT / FIXED INPUT

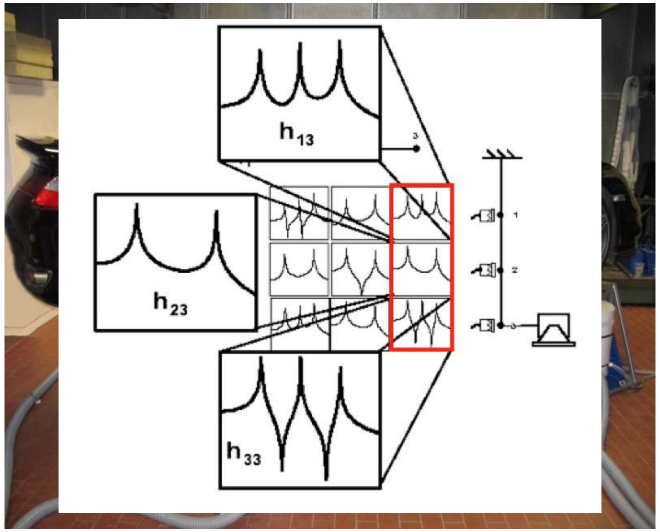
Roving input :

- vary input location, have fixed output locations
- Modal hammer approach

Fixed input :

- Have fixed input location, typically a shaker
- Consider multiple output locations:
  - Multiple accelerometers
  - Laser Doppler Vibrometer (LDV)
  - *Image based techniques (e.g. Digital image correlation)*





INPUT MEASUREMENTS (F)

ROVING INPUT : IMPULSE HAMMER

- Modal or impulse hammer is a instrumented hammer
- Typically relies on Piëzo-electric technology
- Limited hardware required
  - 1 Hammer & 1 accelerometer (more is possible)
- Flexible, suitable for many applications
  - Different sizes
  - Different heads
- Some ‘handiness’ required to avoid *Double impacts*
  - *But on the other hand is that really an issue?*

Large number of points, fast assessment

INPUT MEASUREMENTS (F)

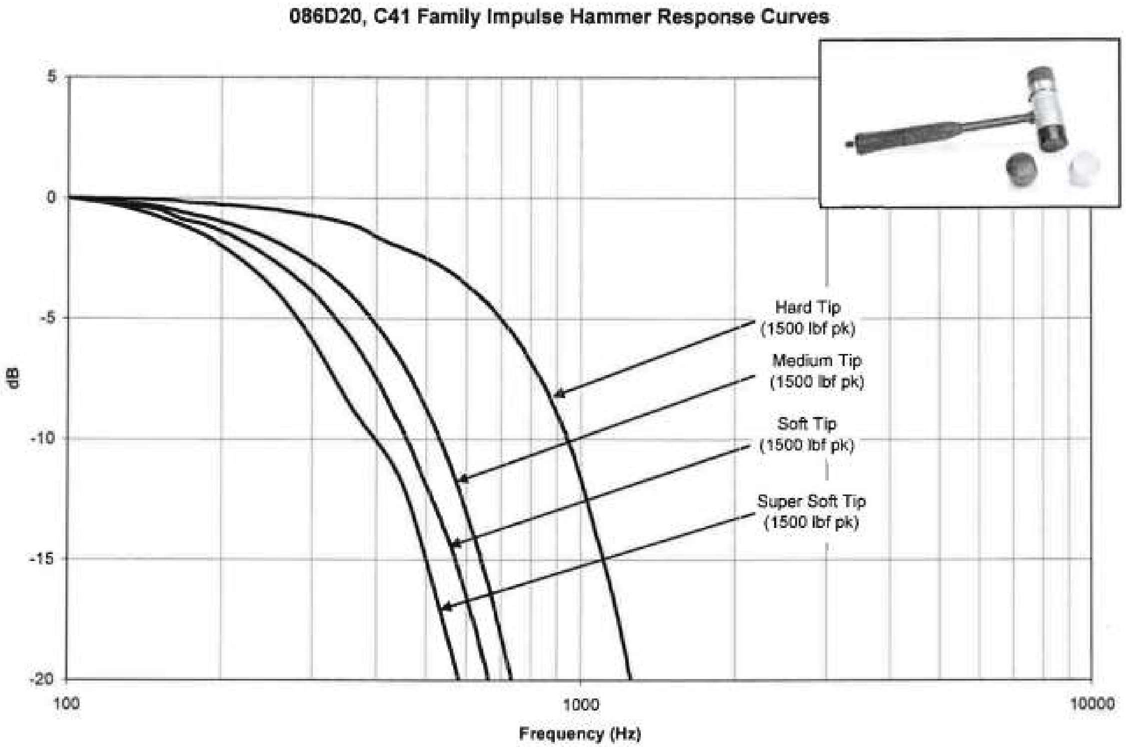
IMPULSE HAMMER : DIFFERENT HEADS

Different materials of the head allow to have different frequency spectra to excite

The softer the head, the longer the impact duration and smaller the bandwidth of the excitation.

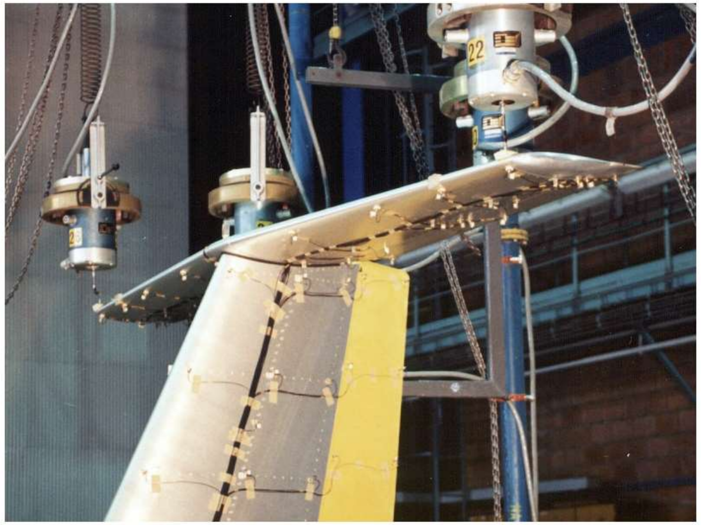
Why?

- More energy in the band of interest
- Avoid issues, i.e. non-linearities at higher frequencies



INPUT MEASUREMENTS (F)

SHAKER



- Typically electrodynamic shaker (can be hydraulic for demanding sizes)
- More controlled input excitation, deterministic. Can basically be any signal that you like (**and we will get back to this!**)
- Setup consists out of one or more shakers, and then as many output measurements as you like.
- Can be done at different sizes, but you can imagine the cost

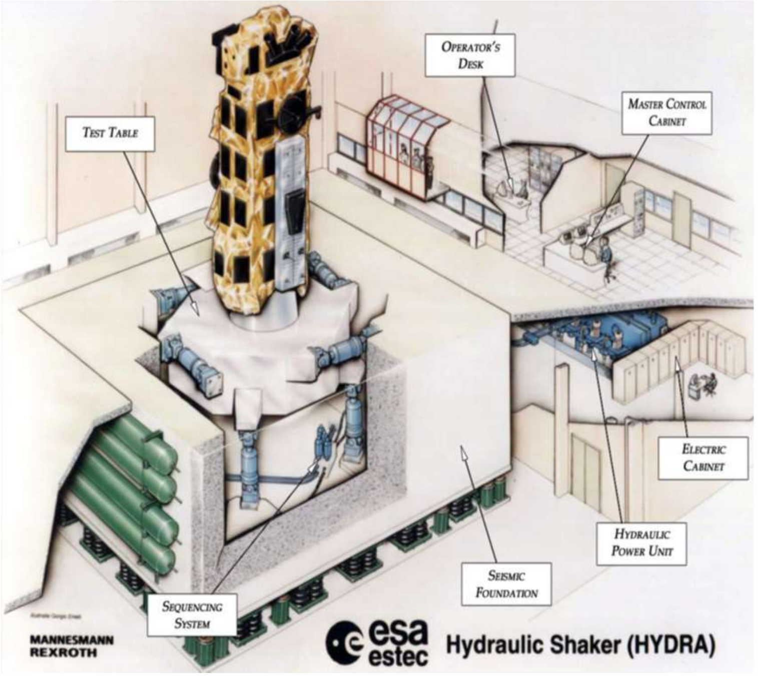
INPUT MEASUREMENTS (F)

SHAKER : IN EVERY SIZE

Example the HYDRA

ESA (ESTEC)

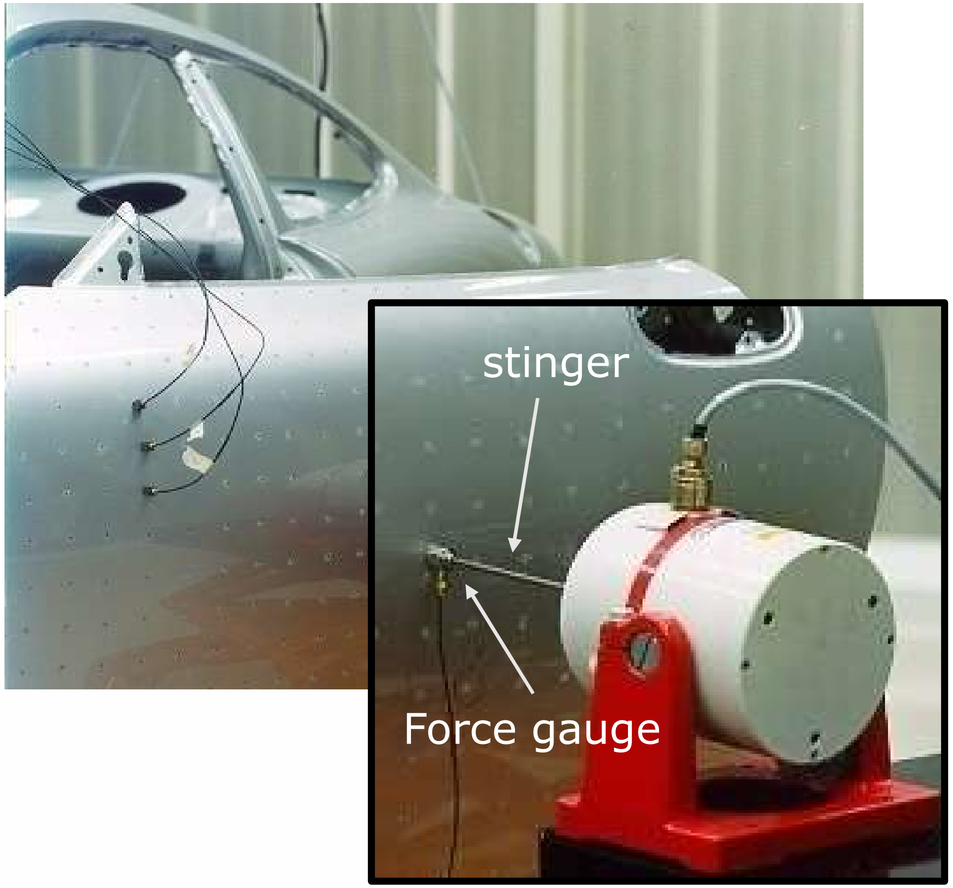
A 6DOF shake table



INPUT MEASUREMENTS (F)

SHAKER

- The shaker is connected to the structure using a slender rod, so-called ‘stinger’
  - Ensure that the excitation is only along the shaker excitation direction
    - High axial stiffness
    - Low transvers and bending stiffness
- Actual force measured using force gauge/cell
- Multiple shakers can be used
  - Energy distribution over (larger) structures
  - 3D – excitation in X,Y,Z direction
  - Multi-reference measurements



INPUT MEASUREMENTS (F)

MODAL HAMMER OR SHAKER

Hammer

- Low cost : hammer + 1 accelerometer
- No physical connection = no dynamic interaction
- Only single accelerometer adding mass
- Only impact /impulse inputs
- Short setup time, experiment time proportional to the number of locations
- Poor for non-linear structures

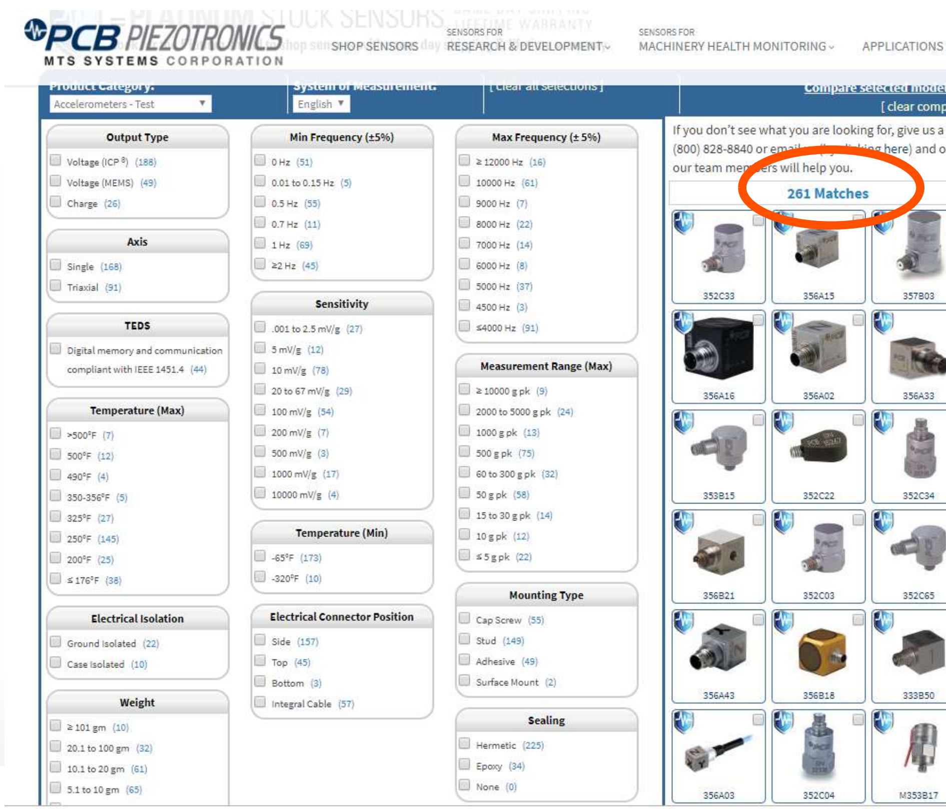
Shaker

- High cost : expensive shaker + multiple sensors
- Shaker is connected to the system, can influence dynamics.
- Added mass effect of the large number of accelerometers
- Controlled input, can be designed for optimal properties. Improved repeatability and Signal to Noise ratio
- Longer setup time, shorter experiment time. Once installed bigger range of tests
- Can be used for non-linear structures

OUTPUT MEASUREMENTS

ACCELEROMETER

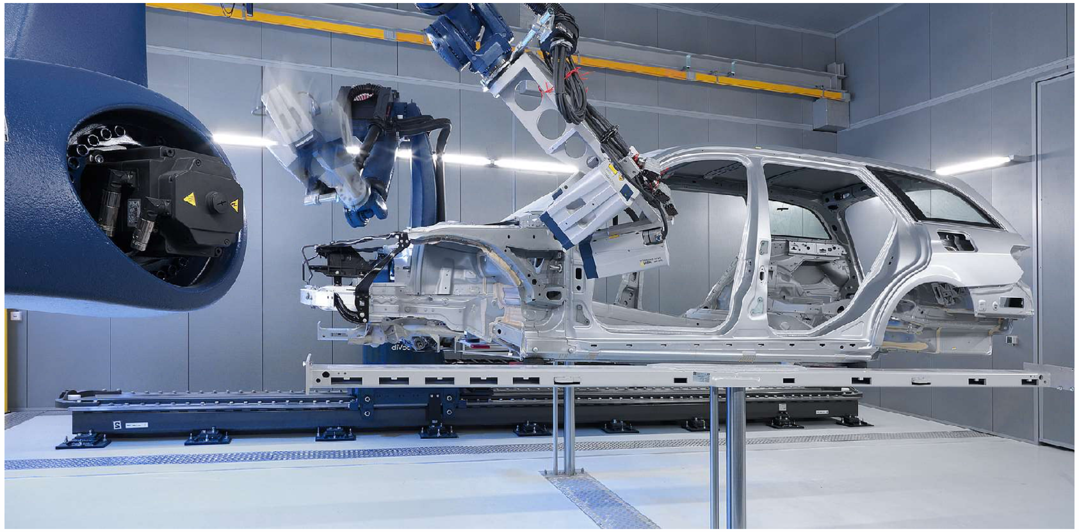
- Three most common technologies available
  - MEMS : more widespread technology, also in e.g. phones
  - Piëzo – electric (ICP) : typical for lab experiments
  - *Force balance : Very sensitive, but bulky and expensive typical for civil applications*
- General specifications to consider :**Does it match my application?**
  - Frequency band : Does it match my application
    - For ICP *Rule-of-Thumb* the bigger the sensor the better its performance at lower frequency bands
    - MEMS can go to 0Hz, thus can measure gravity, from tri-axial you can determine orientation
  - Tri-axial or not?
  - Sensitivity and/or Signal to Noise ratio



OUTPUT MEASUREMENTS

LASER DOPPLER VIBROMETER

- Alternative to the accelerometer, does not introduce added mass
- Measures vibration based on the Doppler effect
- Can be used as a roving output measurements by pointing the laser at different points on the structure
  - Fixed Input – Roving Output



DESIGNING A MODAL TEST

THE PERFECT SETUP

How would we define the best setup?
Well ideally the modal test setup should not alter the structural dynamics of interest.

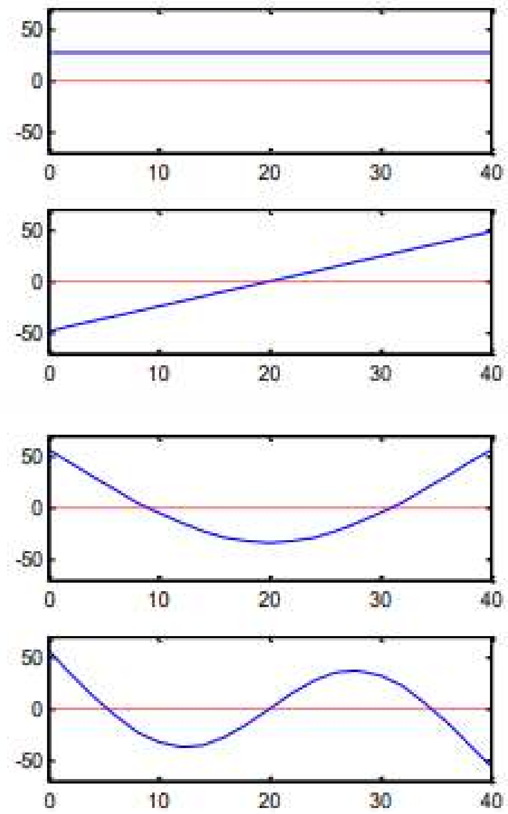
DESIGNING A MODAL TEST

THE IDEAL TEST

Consider our structure is a simple beam *floating in space*

The modes of this system are

- The 6 Rigid body modes (X,Y,Z, Yaw, Roll, Pitch)
- The structural dynamics modes (a.k.a. flexible modes)
  - Infinite number
  - These are of interest



DESIGNING A MODAL TEST

THE IDEAL TEST

Off course we can not have our specimen floating in space, it has to be mounted in a way.

The way the item is suspended will have an impact on the dynamic behavior.

The more rigid, the more the test setup becomes a ‘boundary condition’

Preference for a ‘Free-Free’ setup, as close as possible to ‘floating in space’

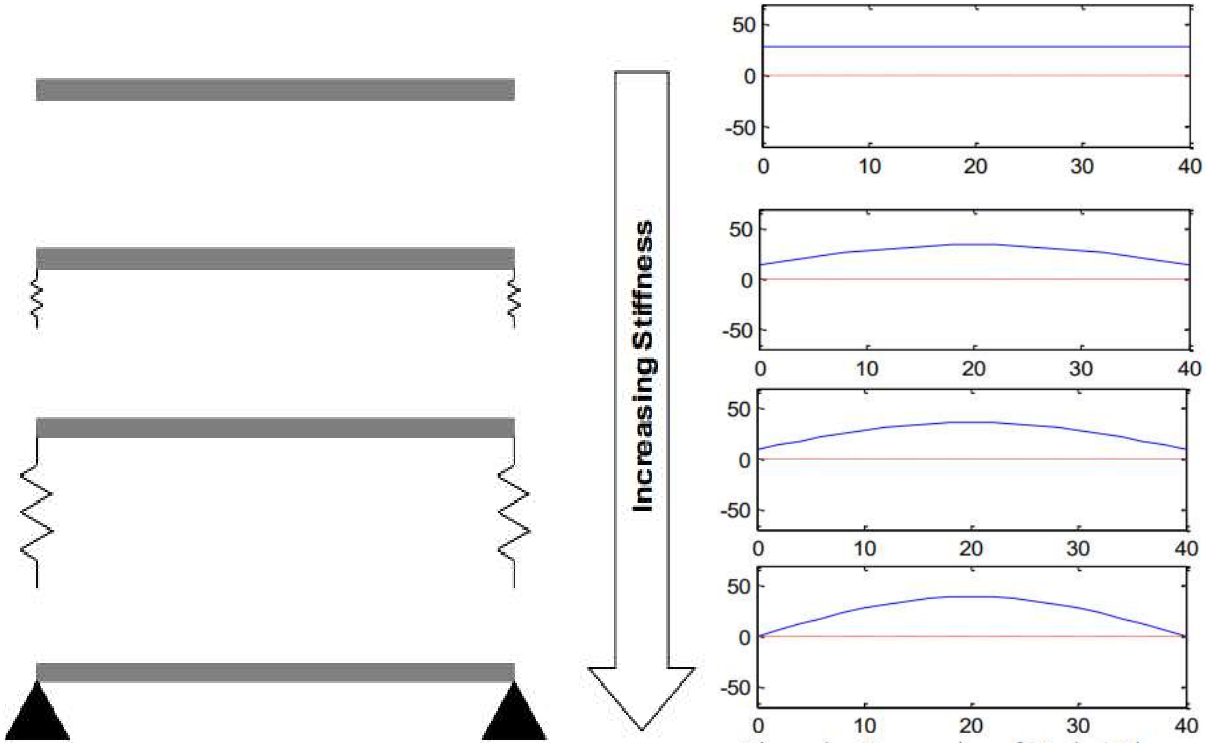
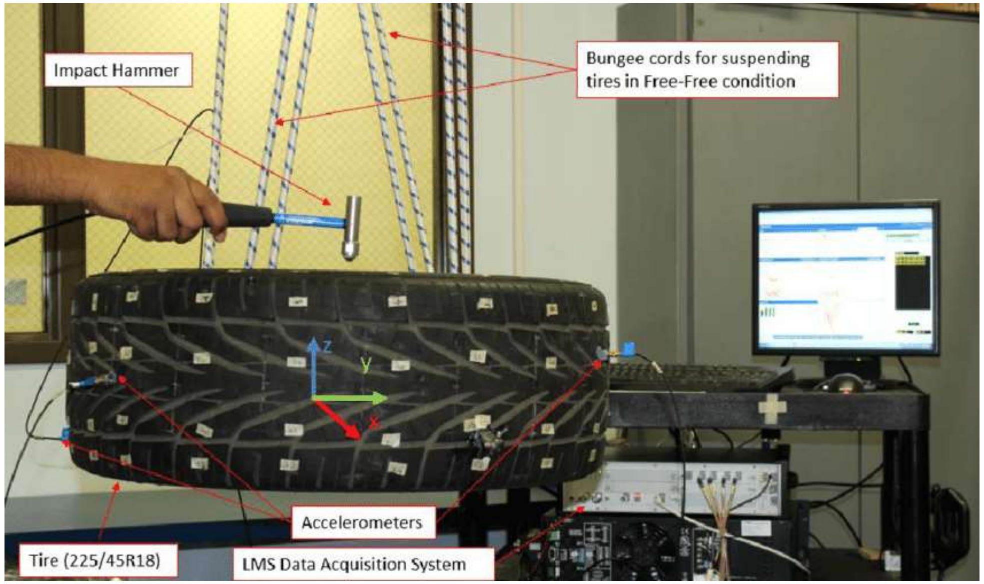
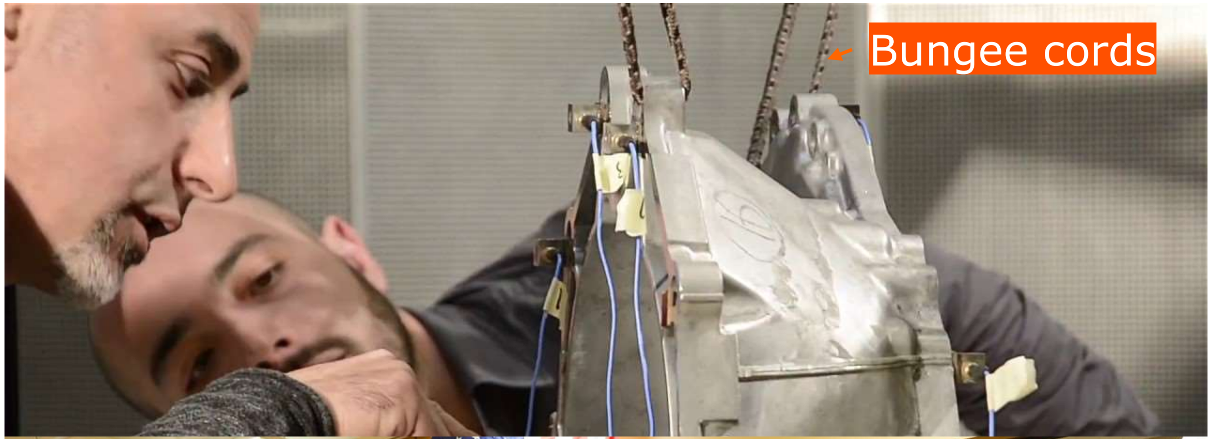


Figure 3 – Progression of Mode 1 Shape

DESIGNING A MODAL TEST

FREE FREE SETUP





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DESIGNING A MODAL TEST

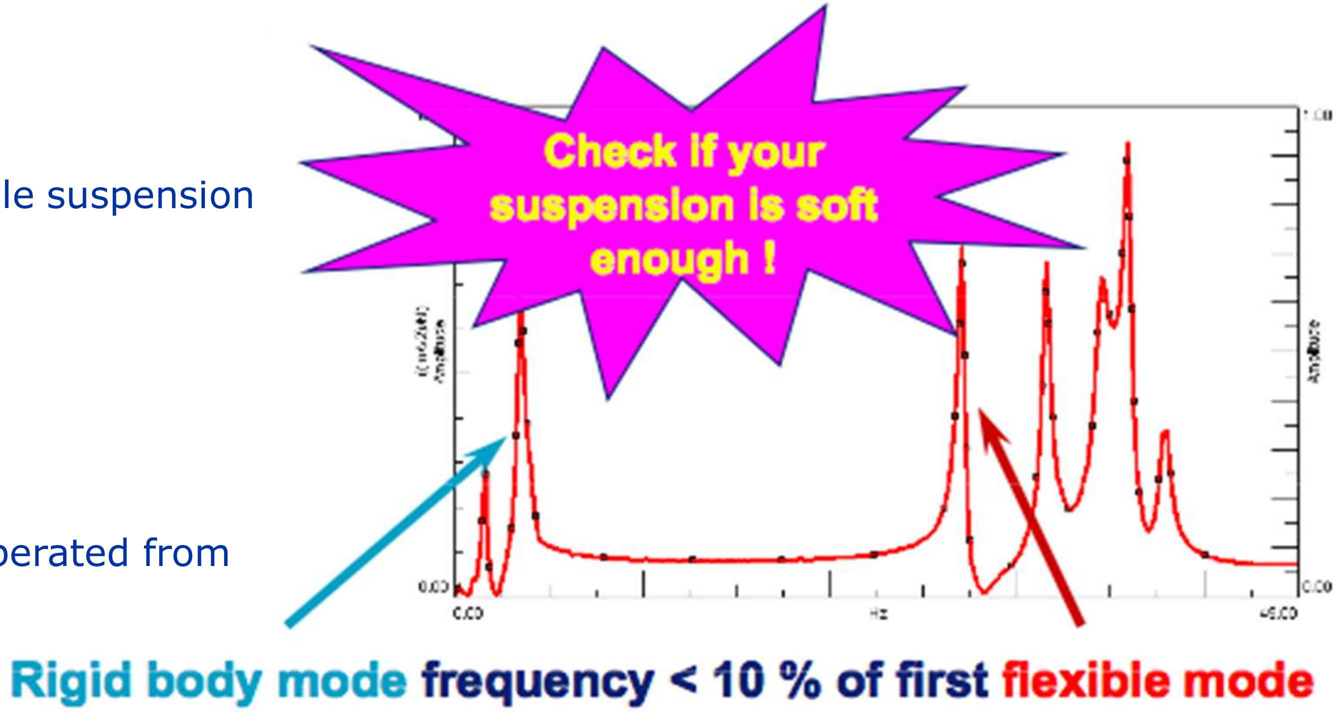
FREE FREE SETUP

Typically (almost) achieved using

- Soft springs, elastic cord, long flexible suspension
- Soft cushion to support structure

Rule of thumb :

- Rigid body modes should be well seperated from flexible modes



SIGNAL PROCESSING

SETTING UP FOR MODAL TESTING

A typical parameter to be set in Data Acquisition systems (DAQ) is the Sample Frequency (Fs).

The sample frequency should respect **Shannon**'s theorem (Engineers' version) :

Always sample at at least twice the highest frequency of interest

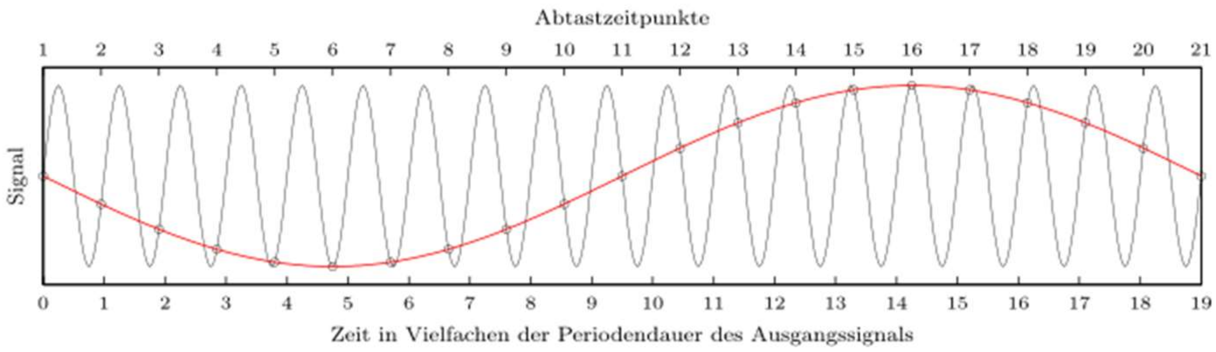
And this to avoid *Aliasing*.

As a safe measure the DAQ will typically also apply an anti-aliasing filter, to attenuate any signal beyond the **Nyquist** (Fs/2) frequency.

SIGNAL PROCESSING

AN EXAMPLE OF ALIASING

In the figure right you can nicely seen that the sampling frequency is far below twice the frequency of the signal of interest.



Classic example : cars' wheels spinning backwards on tv

SIGNAL PROCESSING

SETTING UP FOR MODAL TESTING

Another parameter typical to be set in the configuration are '*windows*'

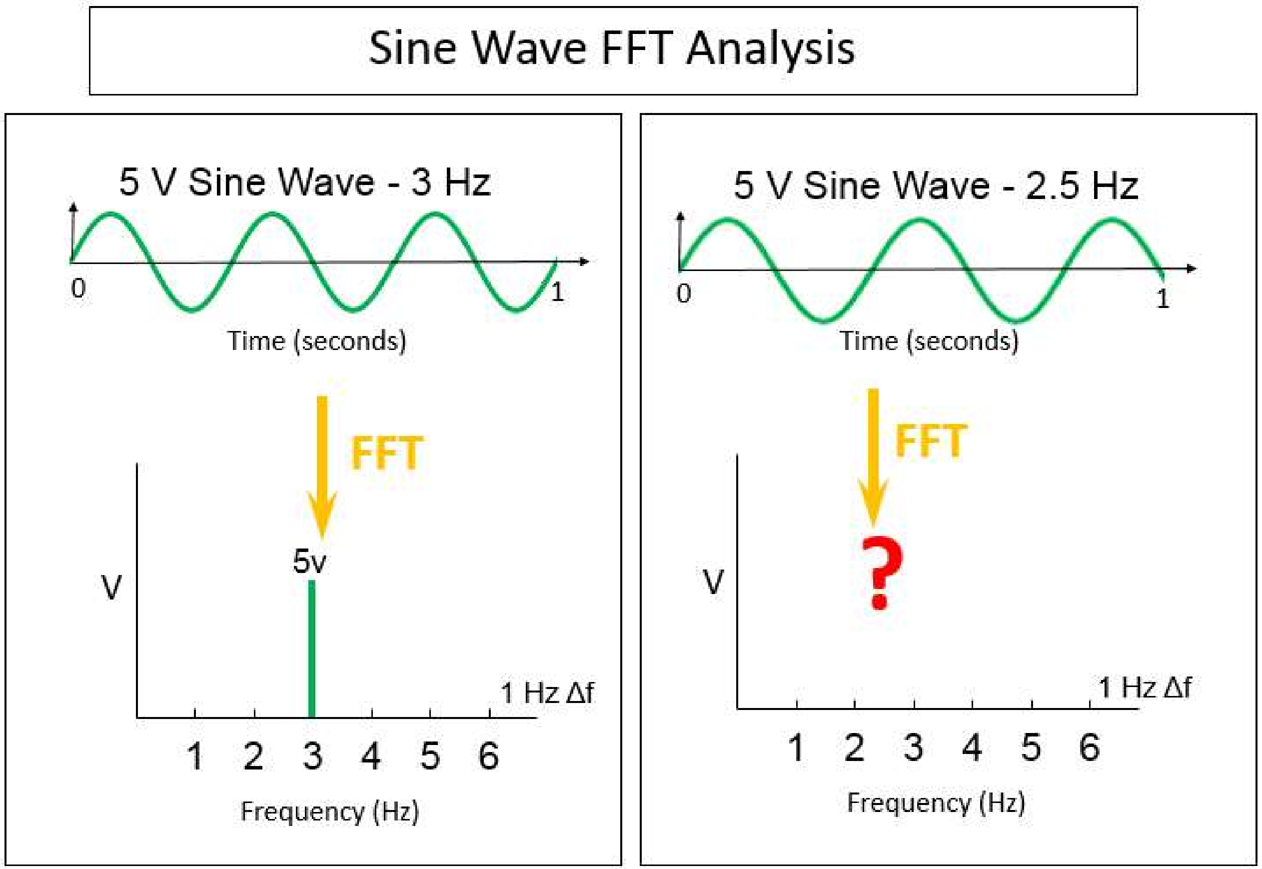
- Window length or measurement time : Defines the frequency resolution
- Window shape

Windows serve to mitigate **Leakage**

SIGNAL PROCESSING

LEAKAGE

Leakage is directly related to calculating the Fourier transformation of a signal of limited duration.



Source : https://community.plm.automation.siemens.com/t5/Testing-Knowledge-Base/Windows-and-Spectral-Leakage/ta-p/432760

SIGNAL PROCESSING

LEAKAGE

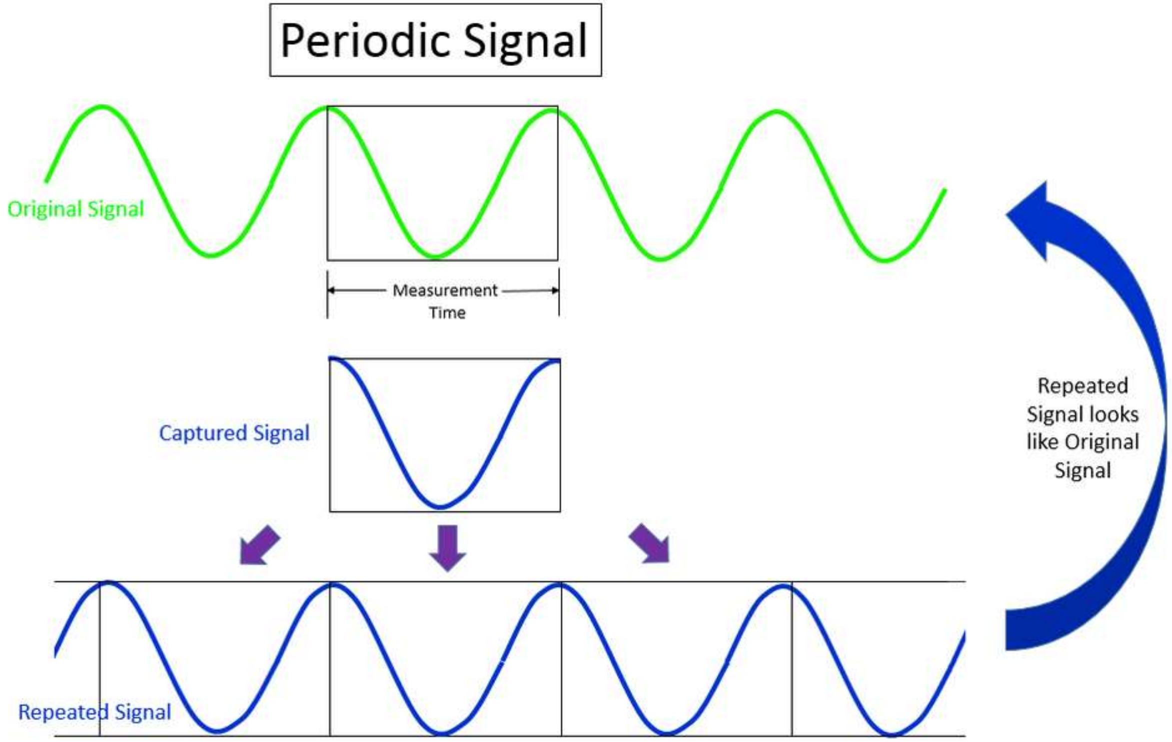
Leakage is directly related to calculating the Fourier transformation of a signal of limited duration.

$$X(f) = \int_{-\infty}^{\infty} x(t) \times e^{-i2\pi f t} \, dt$$

Equation 1: The Fourier Transform has integration limits from negative infinity to positive infinity

The Fourier transform needs to go from −infinity to + infinity.

So we copy paste or short signal



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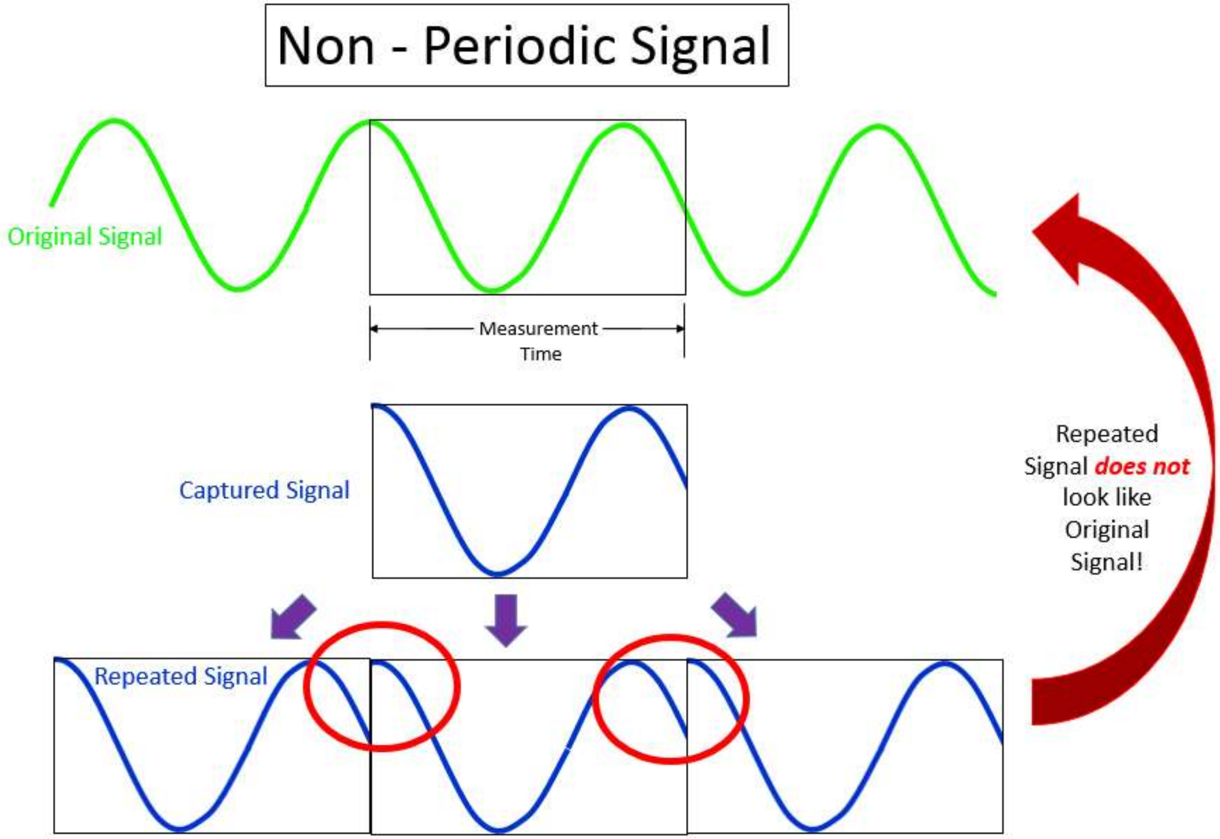


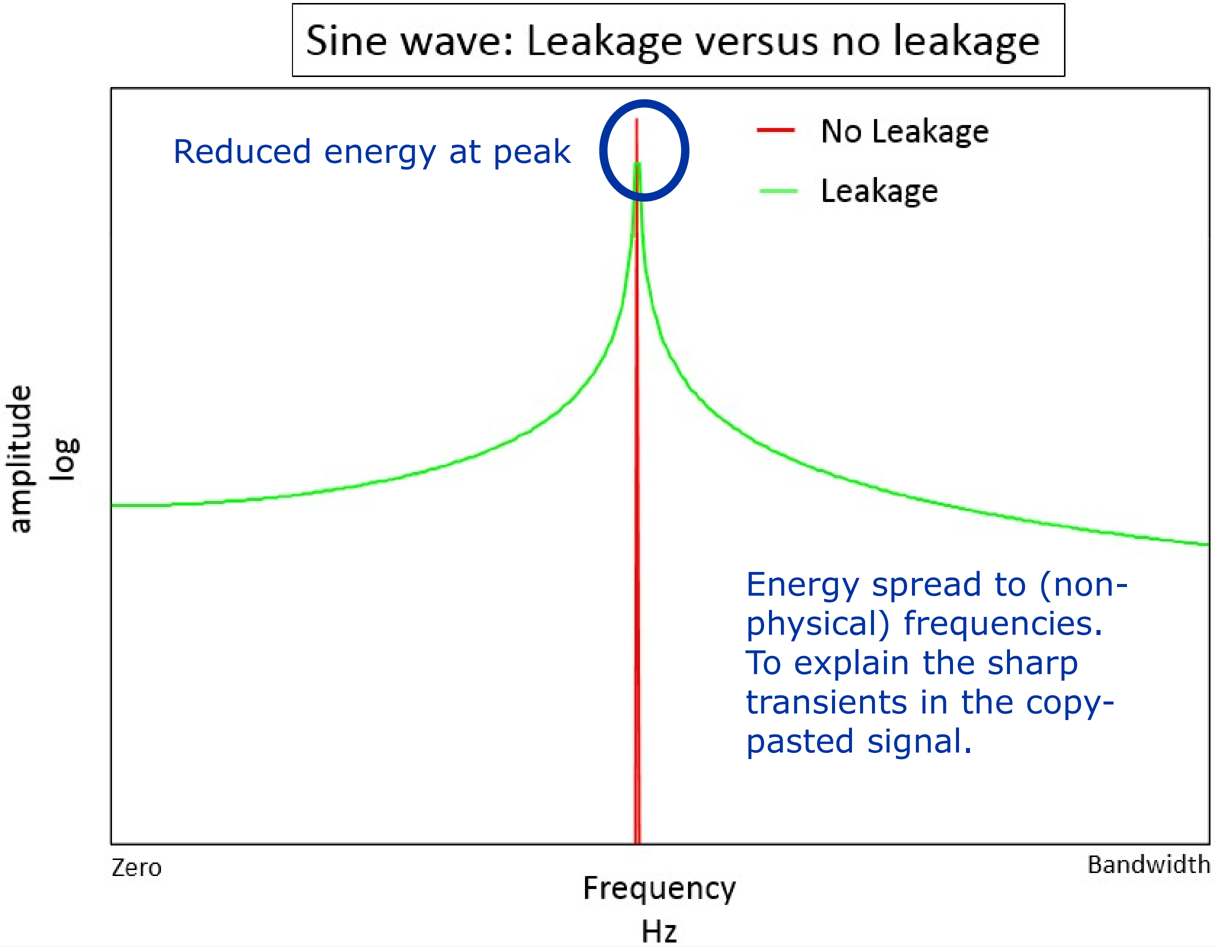
Source : *https://community.plm.automation.siemens.com/t5/Testing-Knowledge-Base/Windows-and-Spectral-Leakage/ta-p/432760*

SIGNAL PROCESSING

LEAKAGE

Leakage causes energy to *leak* into other frequencies, different from the one of interest.

To reduce leakage we need to avoid the sharp transients at the edges of the recorded signal.

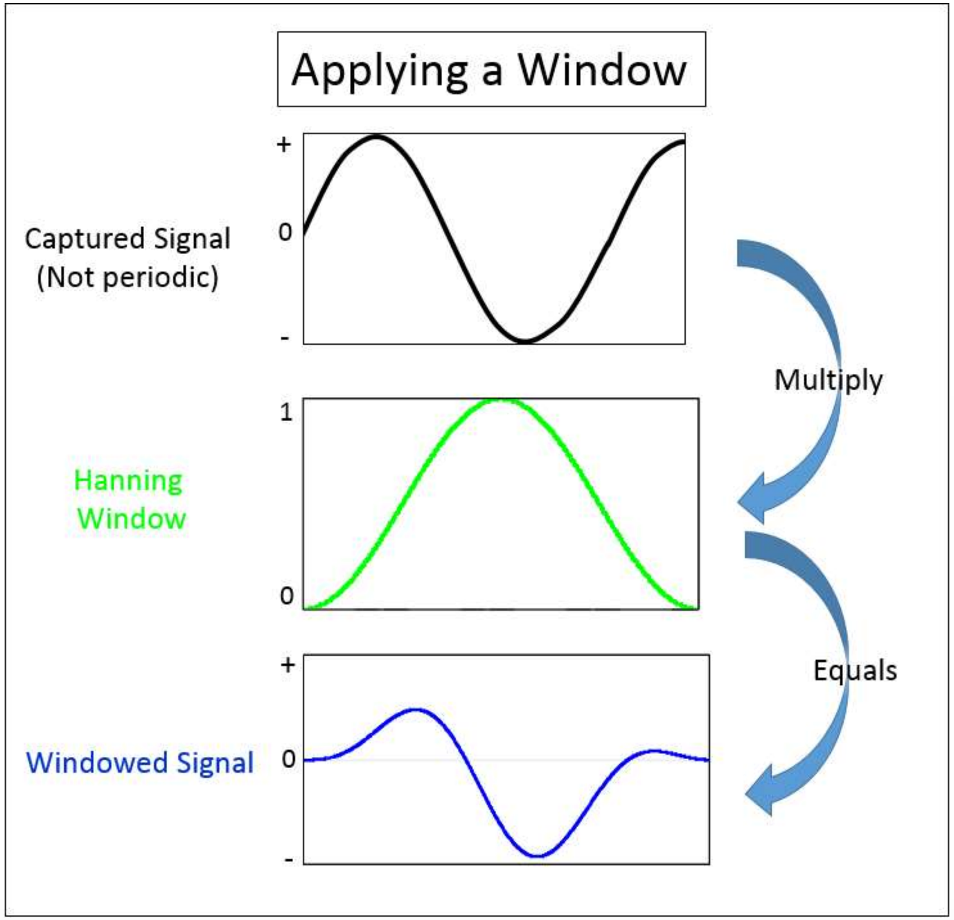


Source : https://community.plm.automation.siemens.com/t5/Testing-Knowledge-Base/Windows-and-Spectral-Leakage/ta-p/432760

SIGNAL PROCESSING

LEAKAGE

Applying a window forces a continuous transition near the ends of the window.



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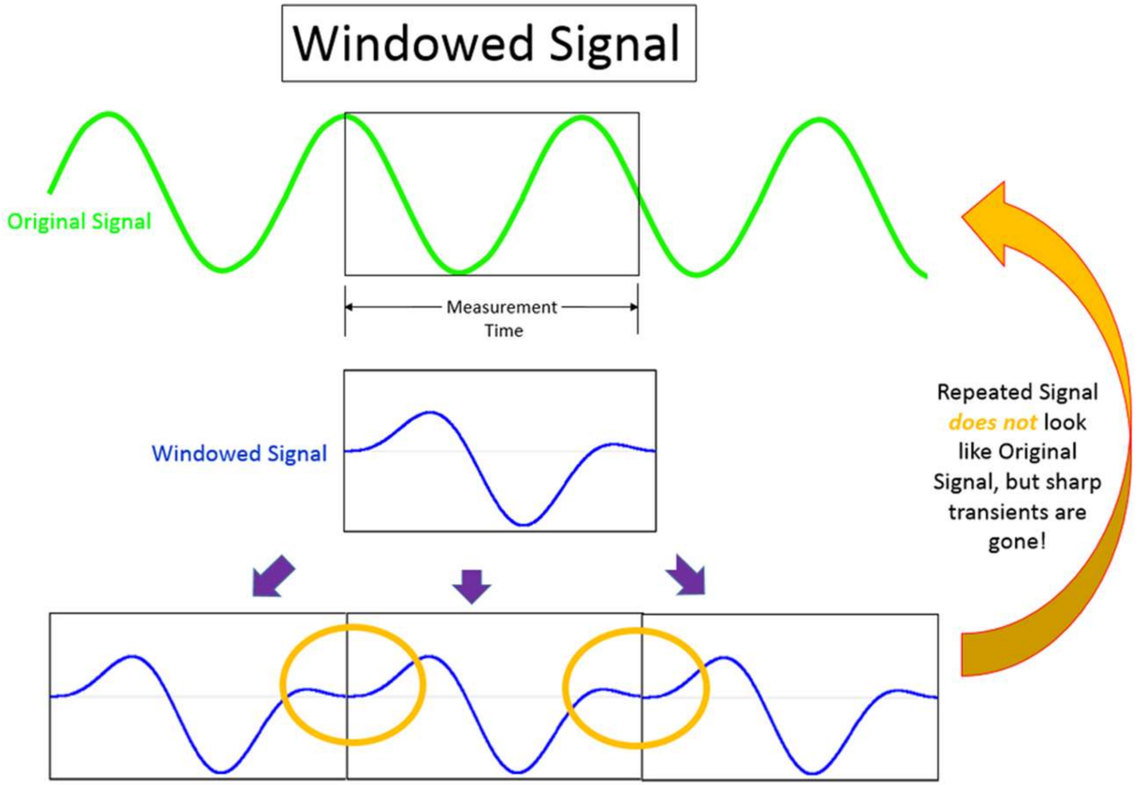
SIGNAL PROCESSING

LEAKAGE

Applying a window forces a continuous transition near the ends of the window.

But this alters my signal as well, often in an irreversible way.



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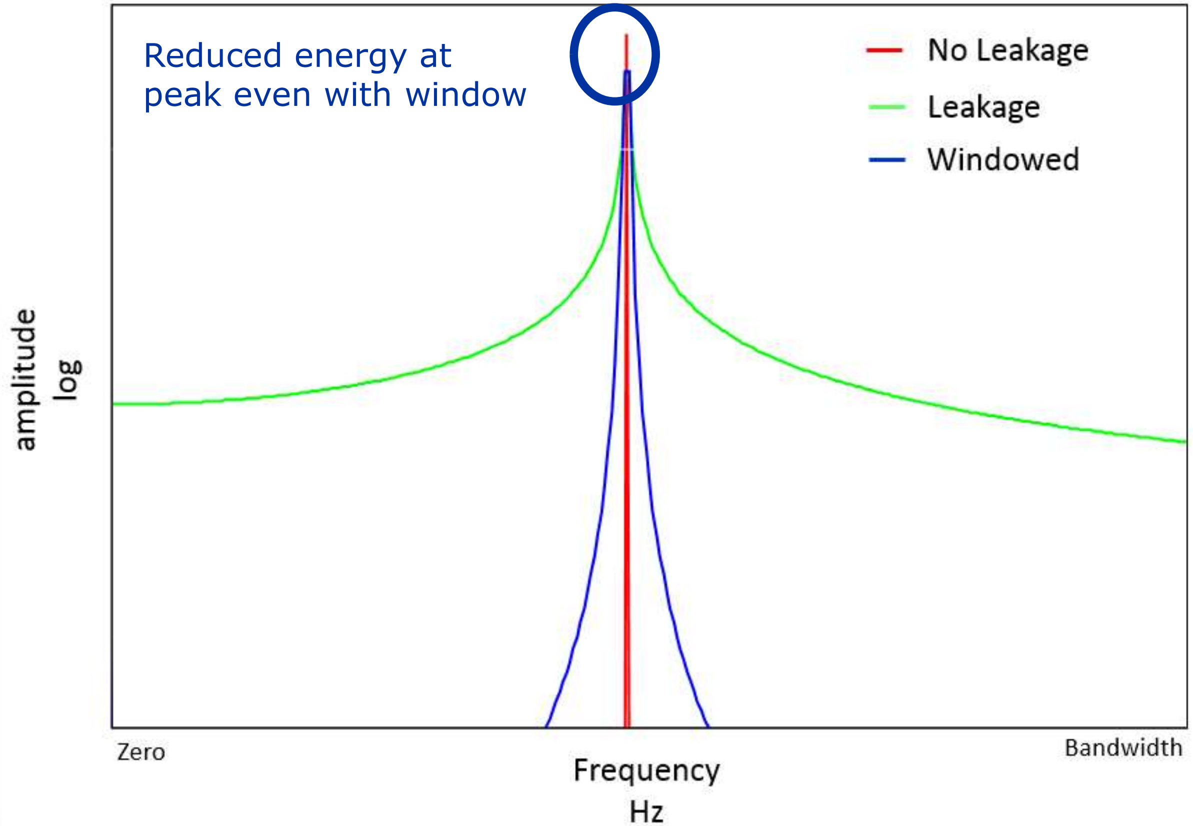
SIGNAL PROCESSING

LEAKAGE

Applying a window forces a continuous transition near the ends of the window.

But this alters my signal as well, often in an irreversible way.

Sine wave: No Leakage, Leakage, Windowed



Source : *https://community.plm.automation.siemens.com/t5/Testing-Knowledge-Base/Windows-and-Spectral-Leakage/ta-p/432760*

SIGNAL PROCESSING

LEAKAGE

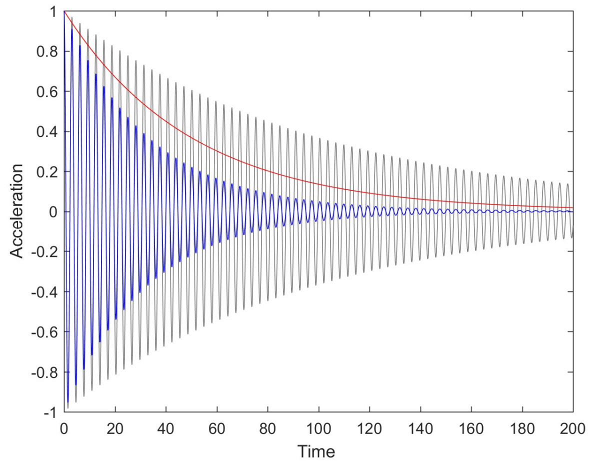
Which window to apply?

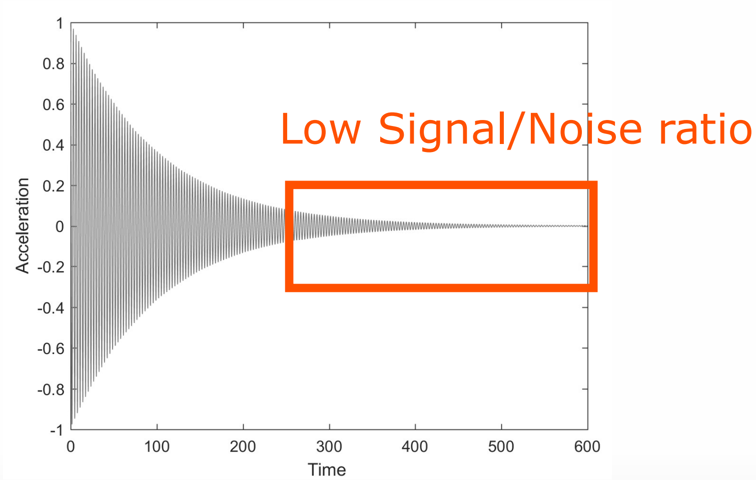
For tests with the modal hammer typically an **exponential window** is used for the output signal.

$$w(t) = e^{-\beta t}$$

The exponential window introduces an additional amount of damping (beta) that actually can be compensated for. The exponential window is as such one of the few windows that can get reversed!

However, as an alternative one could also extend the measurement length.



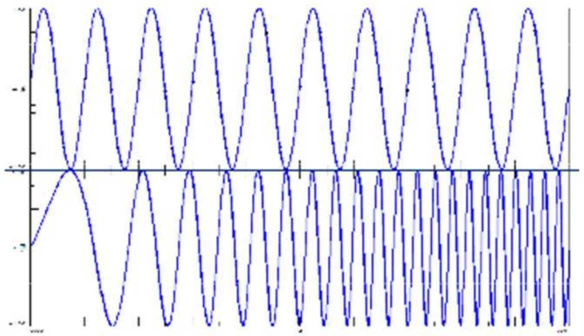


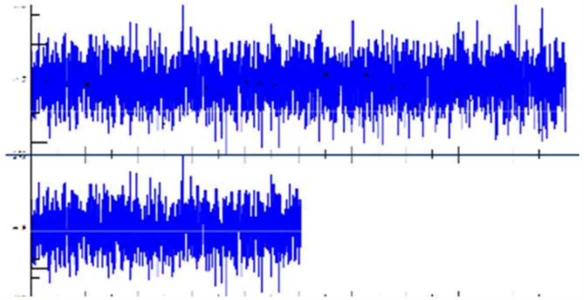
SIGNAL PROCESSING

CHOOSING INPUT SIGNAL FOR THE SHAKER

Typical input signals for shakers are :

- Normal mode excitation : single frequency sine at resonance frequencies
- Stepped or swept sine : sine-wave with varying frequency over time
- Random : i.e. *white* noise
- Burst random : i.e. *white* noise but limited in time





CHOOSING INPUT SIGNAL FOR THE SHAKER

NORMAL MODE

Excite the structure with single sine at resonance frequency, with *tuned* input force combination (typically with several shakers), to have a single mode in resonance.

Oldest method, very accurate but very time-consuming

Get a physical *feel* of the mode

All energy of the shakers goes into a single mode of vibration

Still preferred method in **Ground vibration testing** of aircraft

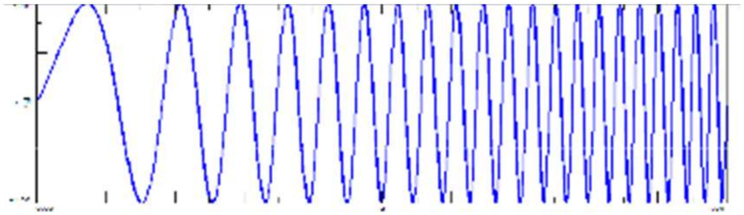
CHOOSING INPUT SIGNAL FOR THE SHAKER

NORMAL MODE

https://www.youtube.com/watch?v=20TN6CGwuKA

CHOOSING INPUT SIGNAL FOR THE SHAKER

SWEPT SINE

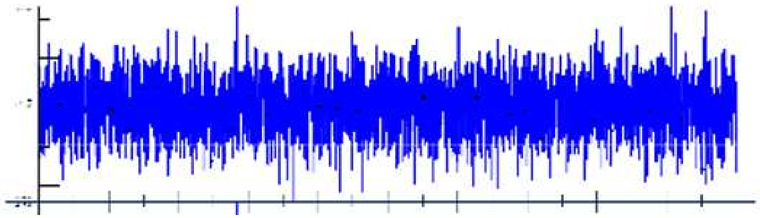


A swept sine (or chirp) is a sine with a continuously varying frequency, it contains all frequencies. A stepped sine is similar, but has discrete frequencies, preferably matching the frequency lines of the final spectrum.

- Still sinusoidal -> All energy in a single frequency at a time, resulting in a better S/N ratio
- Covers entire frequency range -> single test compared to multiple tests in normal mode
- Very controlled signal (known properties) -> Useful to quantify to non-linear behaviour
- Periodical signal -> can be designed to minimize leakage

CHOOSING INPUT SIGNAL FOR THE SHAKER

RANDOM



Random noise is a ‘white noise’ signal and thus covers all frequencies at the same time

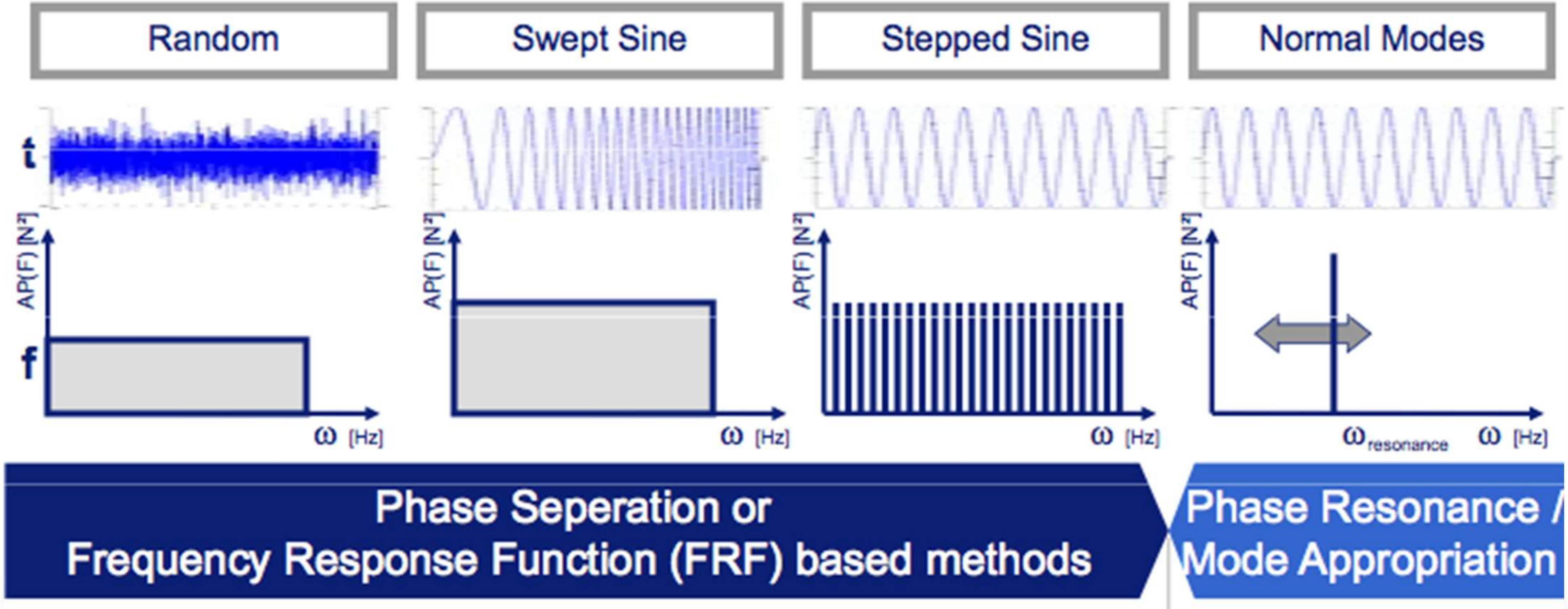
- Most ‘natural’ excitation

- All at once, fast but implies lower energy in each frequency. Might not be sufficient for heavy test subjects.

But… inherently implies leakage.

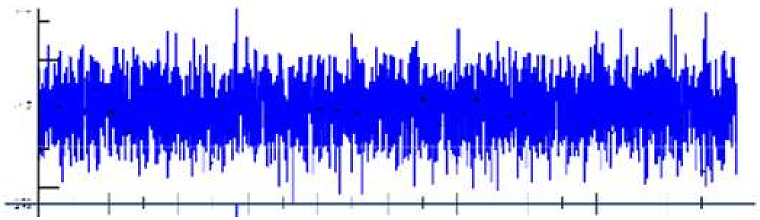
CHOOSING INPUT SIGNAL FOR THE SHAKER

SPECTRAL CONTENT



CHOOSING INPUT SIGNAL FOR THE SHAKER

RANDOM



Random noise is a ‘white noise’ signal and thus covers all frequencies at the same time

- Most ‘natural’ excitation
- All at once, fast but implies lower energy in each frequency. Might not be sufficient for heavy test subjects.

But… inherently implies leakage.

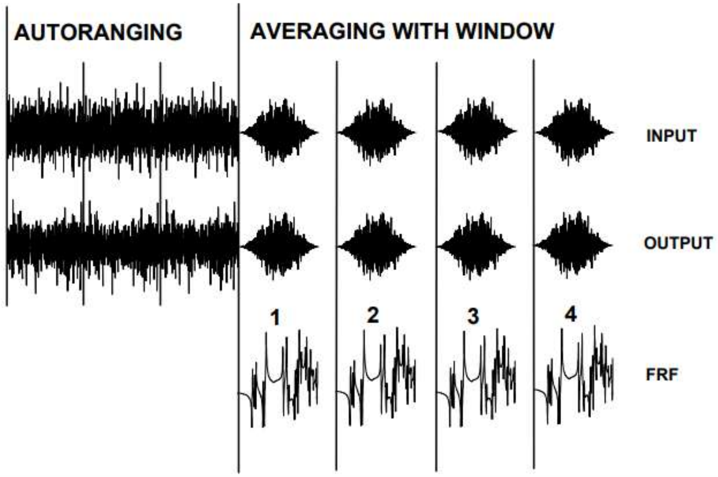
CHOOSING INPUT SIGNAL FOR THE SHAKER

RANDOM

Using Random implies the need for a window as there is no guarantee that the signal at the windows outer edges are continuous.

Typically the **Hanning** window is applied to both input and output.

Hanning window inherently distorts the signal, the effect is irreversible!



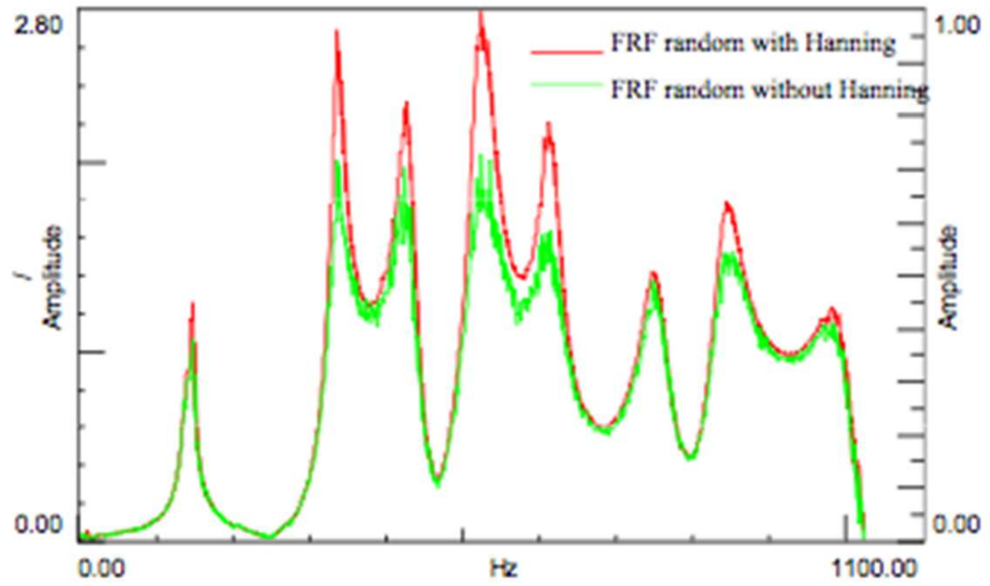
CHOOSING INPUT SIGNAL FOR THE SHAKER

RANDOM

Comparison of random with and without Hanning window after **40 averages**.

Applying the Hanning window reduces leakage and results in a better representation of the actual system.

FRF



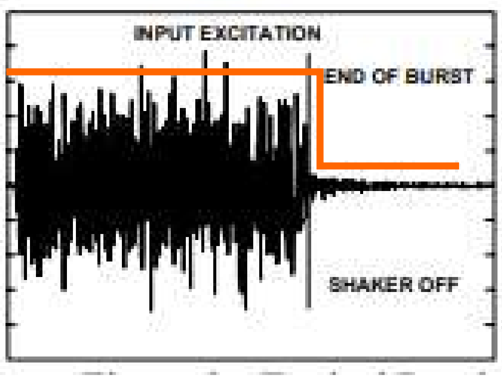
CHOOSING INPUT SIGNAL FOR THE SHAKER

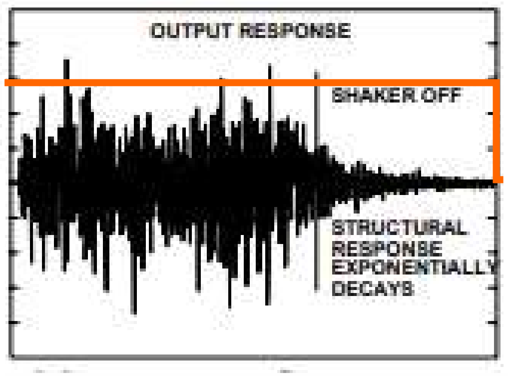
BURST RANDOM

Idea : random signal of limited duration (e.g. 50% of measurement duration)

Allow for signal to decay after burst, natural anti-leakage prevention. But sacrifices S/N ratio

Apply rectangular windows. Limited effect on signal.





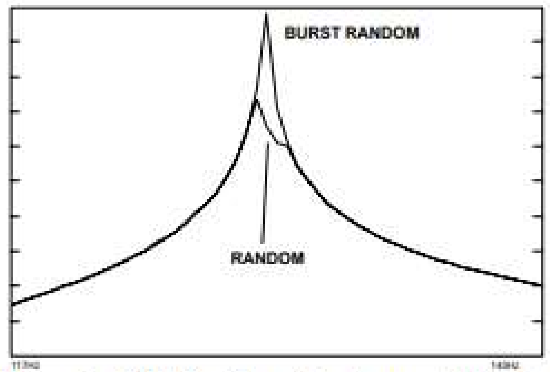


Figure 6 - FRF for Burst Random and Random

CHOOSING INPUT SIGNAL FOR THE SHAKER

CONCLUSION

So which signal do we choose? Best to play to the strengths of using a shaker

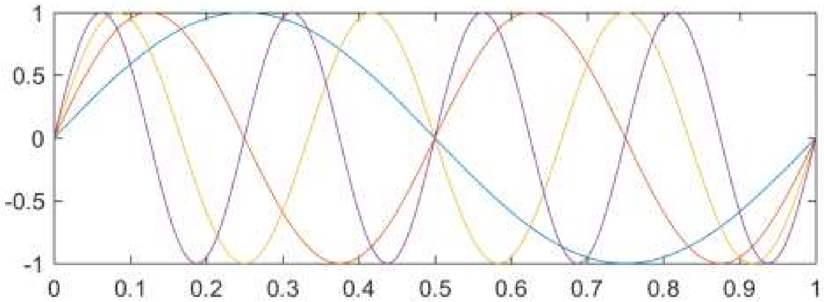
- Easy to repeat experiment
  - Large number of averages is not an issue
- Freedom to choose an optimal signal
  - Replicate real world conditions (when relevant)
  - Avoid leakage, rather than mitigate with windows
- Target energy where we want it to be
  - Heavy structure : swept/stepped sine or normal mode

But maybe none of the aforementioned ones!

CHOOSING INPUT SIGNAL FOR THE SHAKER

EPILOGUE

What if we can engineer a signal that is like random but without leakage!



Summing up sines that are perfectly adjusted to the measurement window.

> No leakage!

CHOOSING INPUT SIGNAL FOR THE SHAKER

MULTISINE

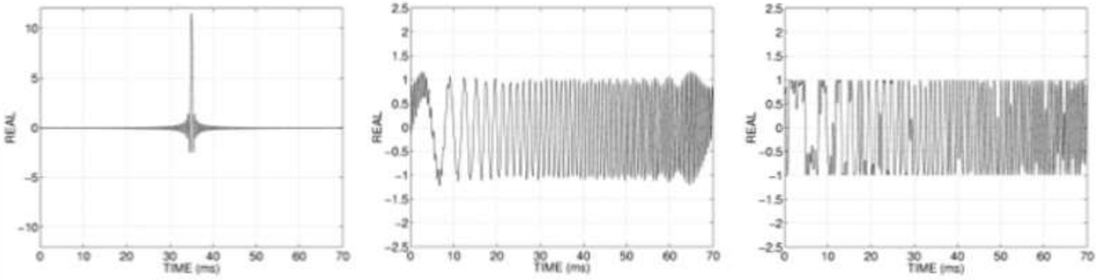
The main variables in the composition of multisines are the **amplitude** and the **phase** of each additional sine.

Signals can be engineered to optimize desired properties.

In the examples to the right this is CF : the Crest Factor.

$$\text{multisine} = \sum_{k=1}^{F} A_k \cos(k\omega_1 \cdot t + \phi_k)$$

$$CF = \frac{PEAK}{RMS}$$



- **Zero phase**
  - **CF = 16**
- **Schroeder phase**
  - **CF = 1.70**
- **L-infinity algorithm**
  - **CF = 1.40**

$\phi_k = 0$

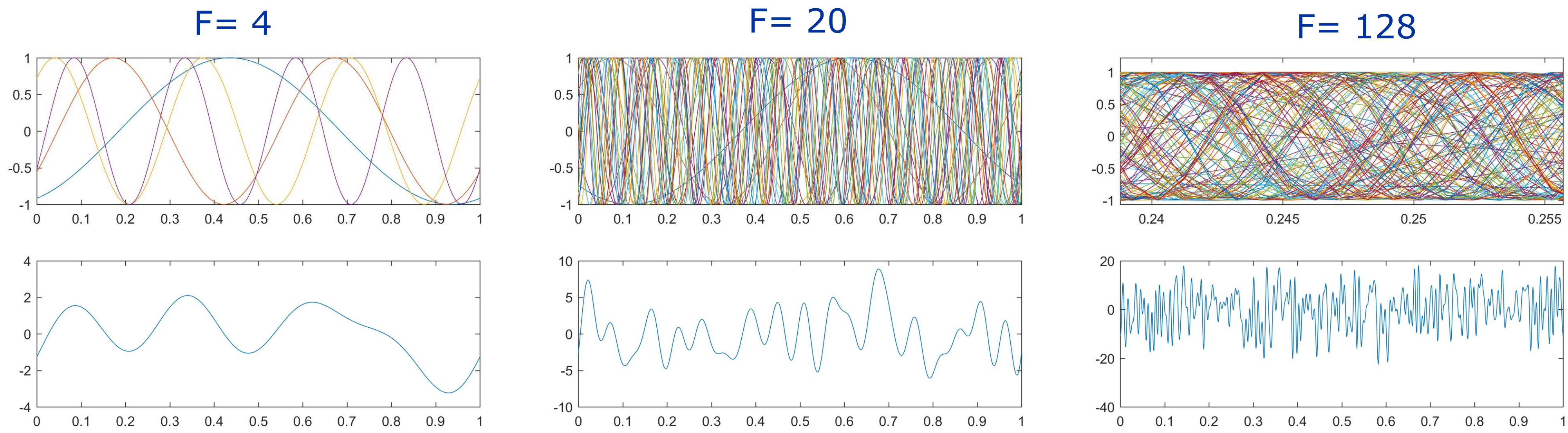
$\phi_k = -\frac{\pi k(k-1)}{F}$

$F = 128,\ A_k = \text{constant}$

CHOOSING INPUT SIGNAL FOR THE SHAKER

MULTISINE

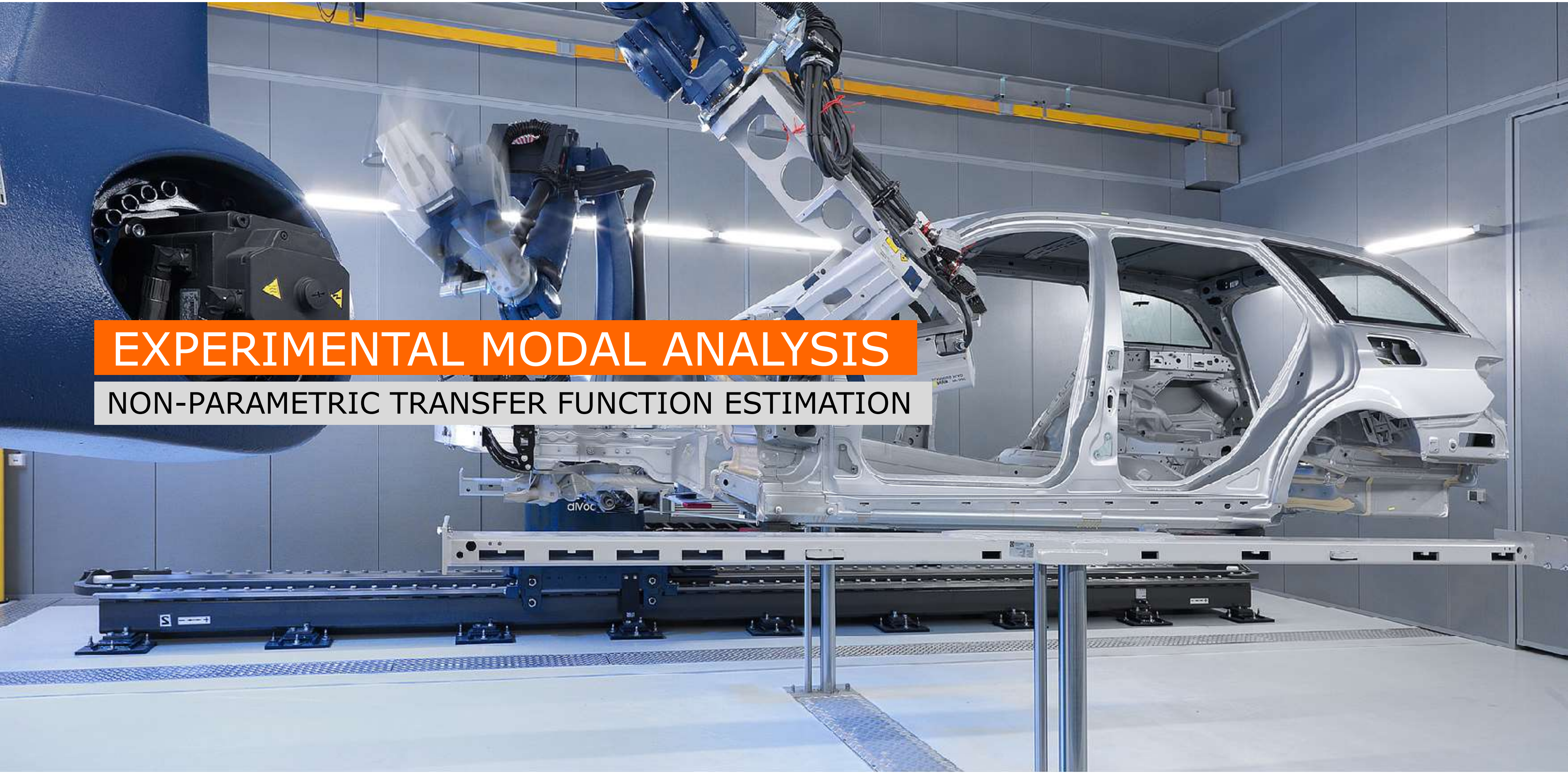
Example using constant amplitude A_k random phase φ_k



Note Zoomed x-scale top figure

EXPERIMENTAL MODAL ANALYSIS

NON-PARAMETRIC TRANSFER FUNCTION ESTIMATION



SO WE DID OUR MEASUREMENTS

…

NOW WHAT?

SIGNAL PROCESSING

FROM MEASUREMENTS TO A TRANSFER FUNCTION ESTIMATE

So at this point we measured inputs (F) and outputs (X).

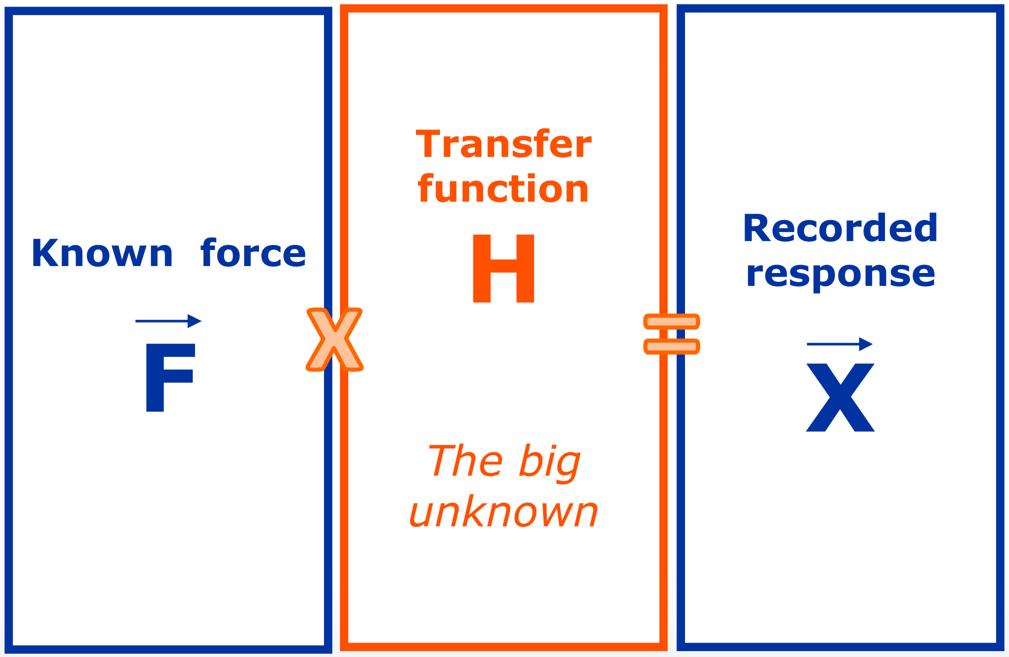
How to compute H? We typically work in the frequency domain

In ideal world

$$H(\omega) = \frac{X(\omega)}{F(\omega)}$$

But that ignores the presence of noise :

- Mechanical noise
- Non-linearities
- Electrical noise in the instrumentation



SIGNAL PROCESSING

FROM MEASUREMENTS TO A TRANSFER FUNCTION ESTIMATE

So first things first, the transfer equation is not a single parameter to be estimated.

A transfer estimate has to be calculated for **every frequency line**

$$H(\omega_i) = \frac{X(\omega_i)}{F(\omega_i)} \quad \forall \omega_i$$

And there is noise on either input, output or both.

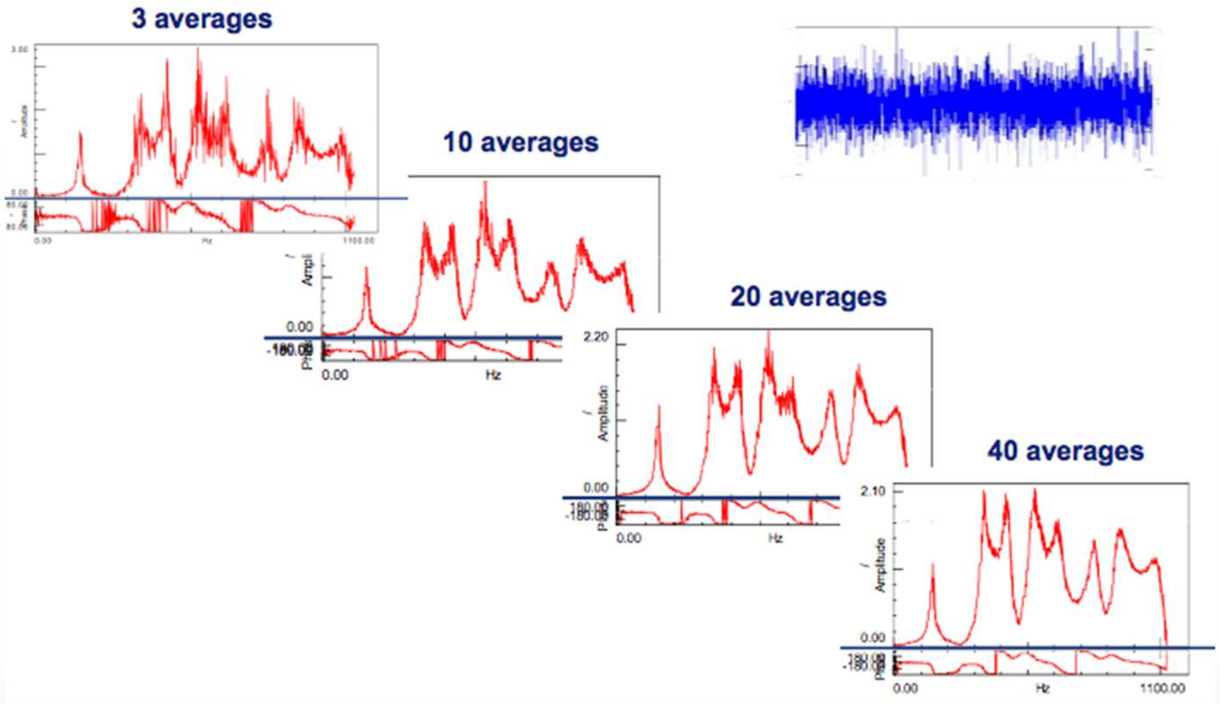
$$H(\omega_i) \approx \frac{X(\omega_i) + n_X}{F(\omega_i) + n_F} \quad \forall \omega_i$$

SIGNAL PROCESSING

FROM MEASUREMENTS TO A TRANSFER FUNCTION ESTIMATE

So we need to average to get rid of (measurement) noise

Note averaging does not resolve leakage!



NON PARAMETRIC ESTIMATE

AVERAGING PREMISE

Let us start with a SISO system and assume that the input force is random.

$$\hat{F}(\omega_i) = \frac{1}{N}\sum_{j}^{N} F_j\,(\omega_i) + \ N_{F,j}$$

But *F* is random with random phase and just like a sine summed with its antiphase signal….

$\hat{F}(\omega_i) \to 0$ when number of averages *N* increases. The same applies to *X*, as the response to random is also random.

$$H(\omega_i) \approx \frac{\sum_j^N X_j(\omega_i) + N_{X,j}}{\sum_j^N F_j(\omega_i) + N_{F,j}} \quad \forall \omega_i$$

Will not work

NON PARAMETRIC ESTIMATE

CALCULATION OF THE POWER SPECTRA

Rather to work with the Spectra themselves, we will work with the power spectra, which will not average to zero (however, phase information is lost)

Auto spectra

$$\widehat{G_{XX}}\,(\omega_i) = \frac{1}{N}\sum_{j}^{N} X_j(\omega_i)X_j^H(\omega_i) \qquad \widehat{G_{FF}}(\omega_i) = \frac{1}{N}\sum_{j}^{N} F_j(\omega_i)F_j^H(\omega_i)$$

Cross spectra

$$\widehat{G_{XF}}(\omega_i) = \frac{1}{N}\sum_{j}^{N} X_j(\omega_i)F_j^H(\omega_i)$$

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NON PARAMETRIC ESTIMATES

H1 ESTIMATE

The H1 estimate uses the cross-spectrum and the input auto-spectrum

$$\boldsymbol{H_1(\omega_i)} = \frac{\widehat{G_{XF}}(\omega_i)}{\widehat{G_{FF}}(\omega_i)} = \frac{G_{XF}(\omega_i) + G_{N_XF}(\omega_i) + G_{XN_F}(\omega_i) + G_{N_XN_F}(\omega_i)}{G_{FF}(\omega_i) + G_{N_FF}(\omega_i) + G_{FN_F}(\omega_i) + +G_{N_FN_F}(\omega_i)}$$

If we now assume that **input and output are not correlated to any noise** and **input noise and output noise are uncorrelated,** than for sufficient number of averages.

$$H_1(\omega_i) = \frac{\widehat{G_{XF}}(\omega_i)}{\widehat{G_{FF}}(\omega_i)} = \frac{G_{XF}(\omega_i)}{G_{FF}(\omega_i) + G_{N_FN_F}(\omega_i)}$$

If we now assume that **no input noise**

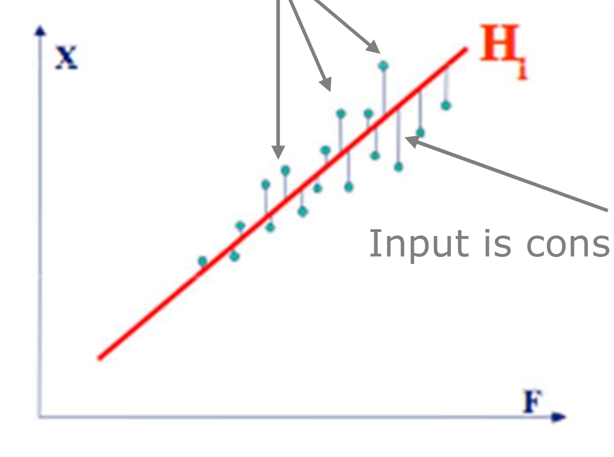
$$H_1(\omega_i) = \frac{\widehat{G_{XF}}(\omega_i)}{\widehat{G_{FF}}(\omega_i)} = \frac{G_{XF}(\omega_i)}{G_{FF}(\omega_i)}$$

NON PARAMETRIC ESTIMATES

H1 ESTIMATE

The H1 estimate is the equivalent of a **Least squares estimate** of the transfer function

Different measurements at a single frequency



Input is considered exact

NON PARAMETRIC ESTIMATES

H1 ESTIMATE

But what if there is noise on the inputs

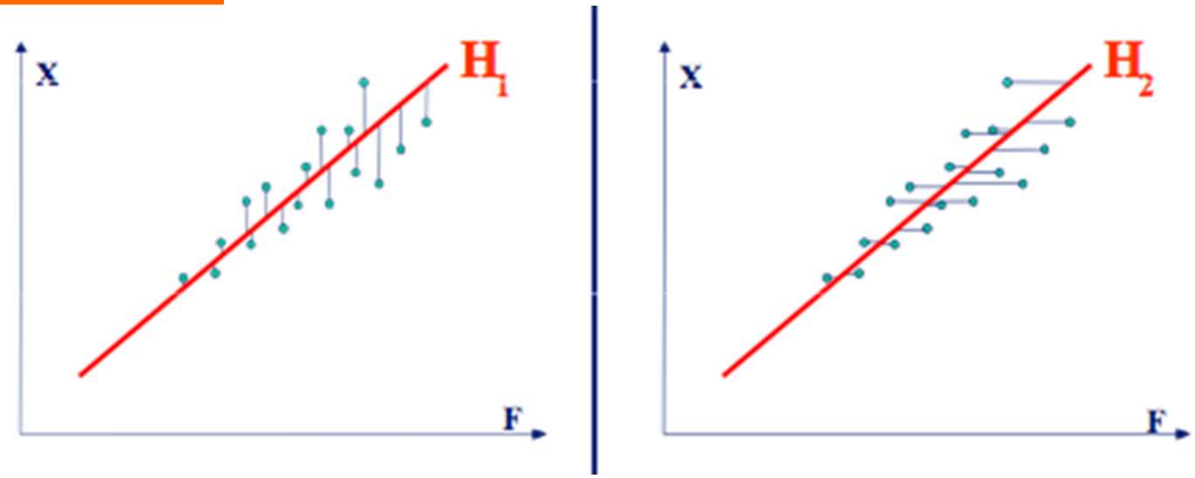
$$\boldsymbol{H}_1(\omega_i) = \frac{\widehat{\boldsymbol{G}_{XF}}(\omega_i)}{\widehat{\boldsymbol{G}_{FF}}(\omega_i)} = \frac{\boldsymbol{G}_{XF}(\omega_i)}{\boldsymbol{G}_{FF}(\omega_i) + \boldsymbol{G}_{N_F N_F}(\omega_i)}$$

Biased outcome, **underestimation** of the transfer function.

This typically manifests itself at frequencies where the amplitude of the force spectrum is low. Often at the resonance frequencies.

NON PARAMETRIC ESTIMATES

H1 AND H2 ESTIMATE

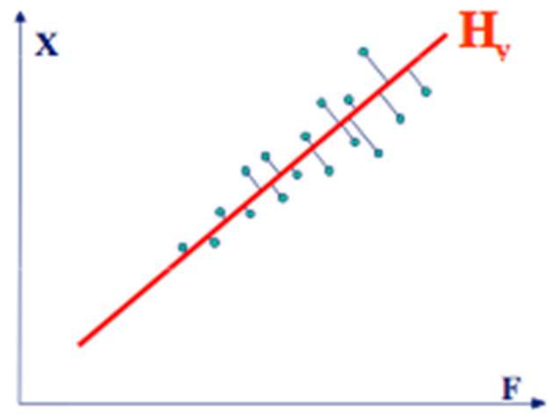


	H1	H2
Formula	$H_1(\omega_i) = \dfrac{\widehat{G_{XF}}(\omega_i)}{\widehat{G_{FF}}(\omega_i)}$	$H_2(\omega_i) = \dfrac{\widehat{G_{XX}}(\omega_i)}{\widehat{G_{FX}}(\omega_i)}$
Assumption	No noise on input (F)	No noise on output (X)
Potential issue	Underestimation in presence of input noise	Overestimation in presence of output noise
Potential gain	Better at anti-resonances	Better at resonances

NON PARAMETRIC ESTIMATES

HV ESTIMATE

In the presence of both input and output noise, the Hv estimator is used. Unlike the H1 and H2, the Hv is a **Total Least Squares** estimator.



"When in doubt use Hv"

NON PARAMETRIC ESTIMATES

EXTENSION TO MIMO

Up to now we considered the system to be SISO making the powerspectra 0D. However, they are 2D when considering multiple input (Ni) and multiple outputs (No).

These are now (No x 1) vectors

$$\overline{\overline{\widehat{G_{XX}}}}(\omega_i) = \frac{1}{N}\sum_j^N \boldsymbol{X}_j(\omega_i)\boldsymbol{X}_j^H(\omega_i)$$

Dimensions of the powerspectra

G_{XX} : (No x No), G_{FF} : (Ni x Ni), G_{XF} : (No x Ni), G_{FX} : (Ni x No)

$$\overline{\overline{H_2}}(\omega_i) = \overline{\overline{\widehat{G_{XX}}}}(\omega_i) \cdot \overline{\overline{\widehat{G_{FX}}}}^{\dagger}(\omega_i)$$

With $\cdot^{\dagger}$ the pseudo inverse

QUALITY ASSURANCE

DRIVING POINT

An important (must-do) measurement is the so-called **Driving point (DP) FRF** this the FRF where input and output location are the same.

- For modal hammer testing : hitting **near** the accelerometer

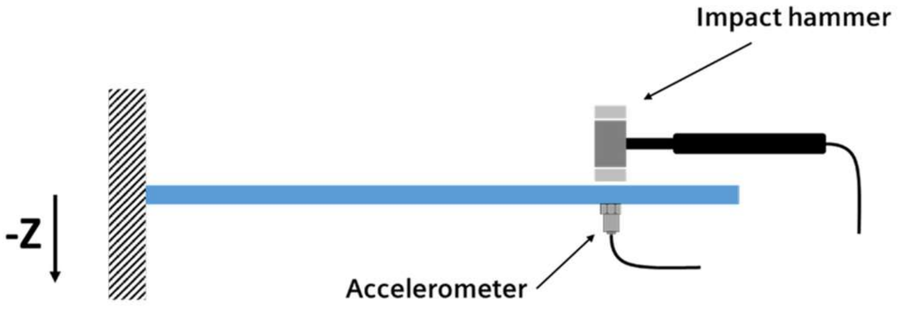
- For shaker testing : having an output measurement near the input location

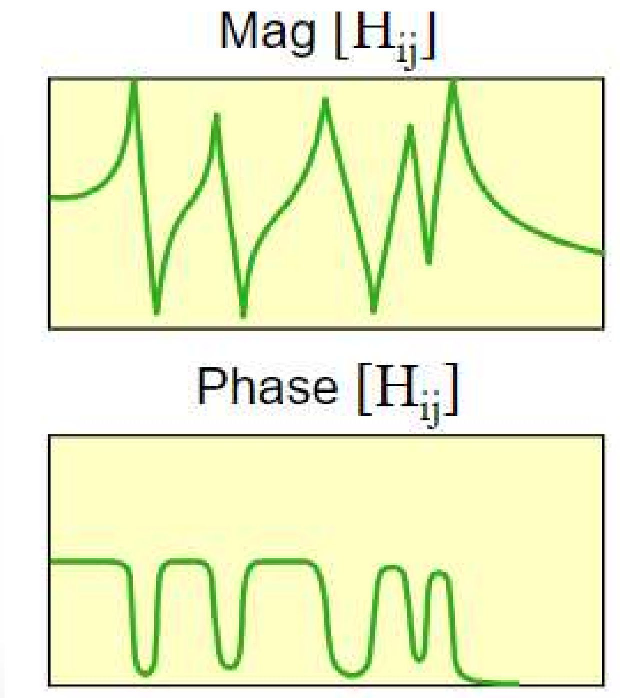
DP should have :

- All modes present -> else you installed the accelerometer or shaker in a nodal point!

- Alternating resonances and anti-resonances

- Phase jumping between 0 – 180 degrees





QUALITY ASSURANCE

COHERENCE

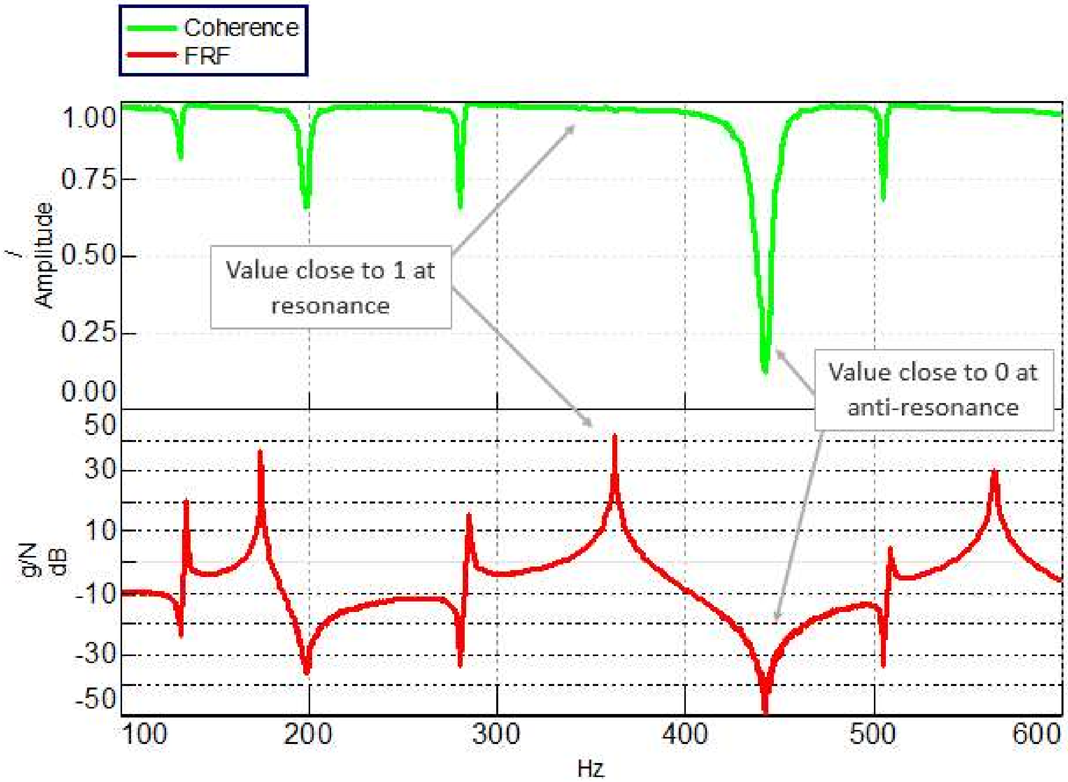
The **coherence** is a great metric to assess the quality of the measurements.

$$\gamma^2 = \frac{|G_{FX}|^2}{G_{XX} G_{FF}}$$

"How well does input and outputs line up (through linear glasses)"

Coherence should be close to 1, else

- Noise in the measurements
  - Variation in excitation direction (e.g. with hammer)
  - Sensor, cabling issues (e.g. overloads, bad connection)
- Non-linearities
- Leakage



Note that if only one measurement is performed, the coherence will be a value of 1! The value will be one across the entire frequency range – giving the appearance of a "perfect" measurement. This is because at least two FRF measurements need to be take and compared to start to calculate a meaningful coherence function. Don't be fooled!

QUALITY ASSURANCE

COHERENCE

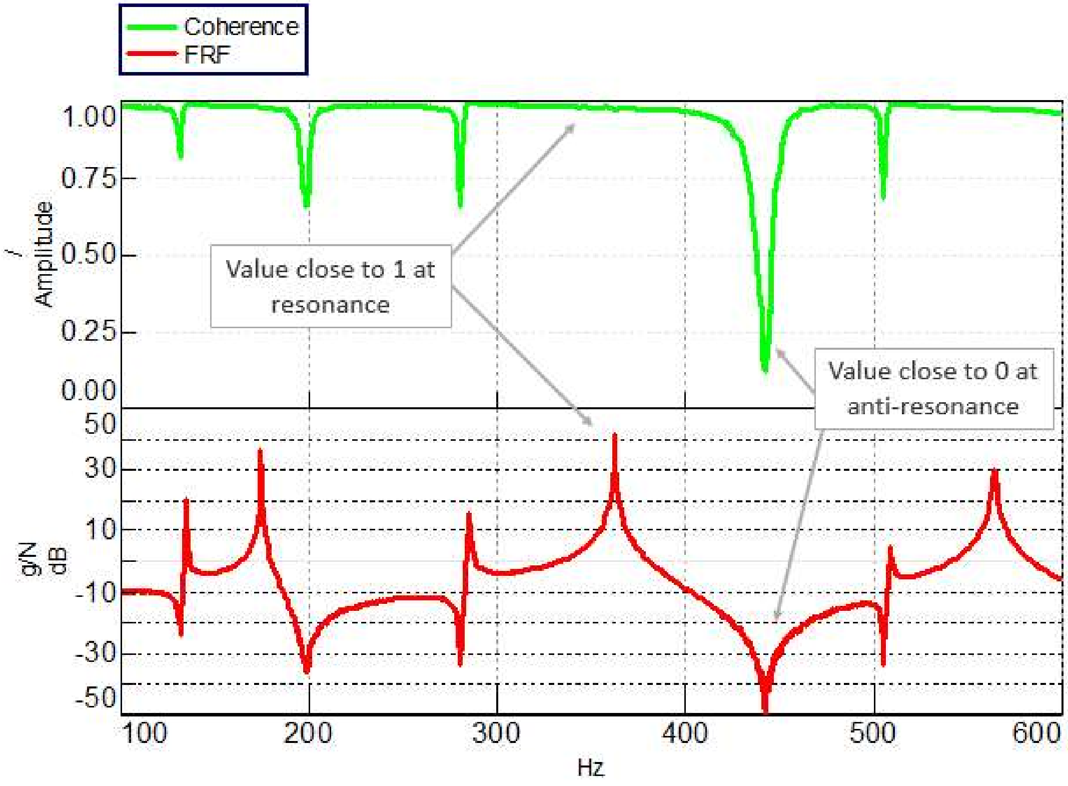
Zero at the anti-resonance?!

This is normal behaviour, the system response is minimal (as expected).

Signal to Noise is poor at these frequencies

-> Input does not match with 'noisy' output

-> Low coherence



QUALITY ASSURANCE

COHERENCE (EXERCISE)

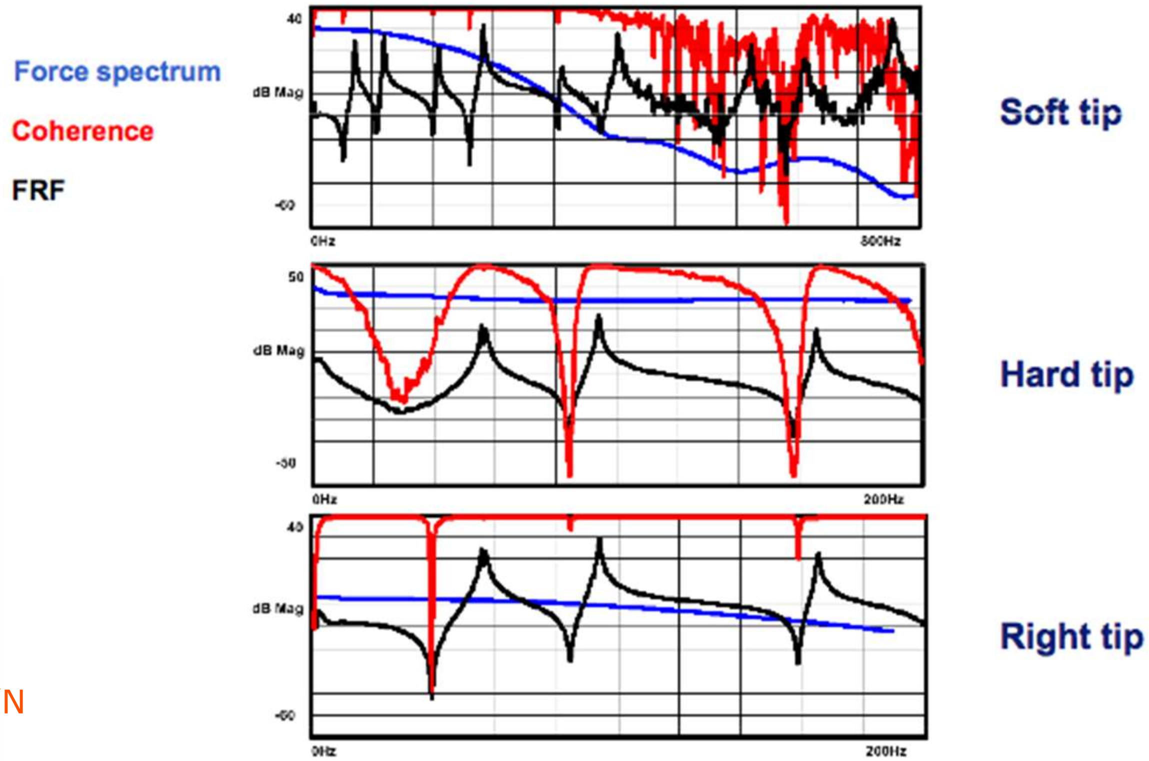
What is the cause of the poor coherence during these impact testing results of :

- High frequency content with the soft tip

  Insufficient energy at high frequencies, poor S/N of the input.

- Area around anti-resonance with hard tip

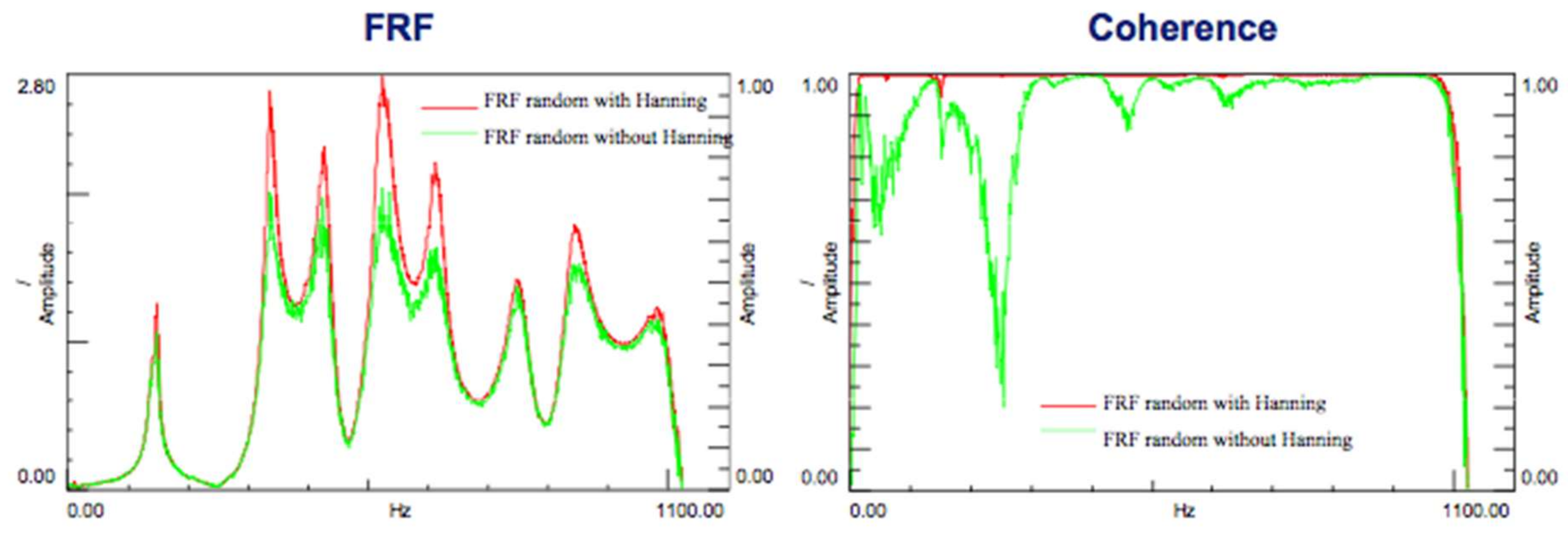
  Insufficient energy injected into the system (short impact). Poor excitation of anti-resonances, poor S/N ratio of output near anti-resonance.



QUALITY ASSURANCE

COHERENCE (EXERCISE)

Impact of leakage is also visible in the coherence



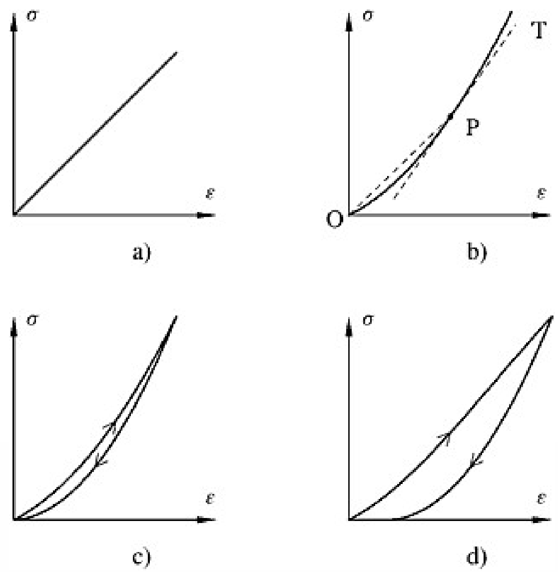
QUALITY ASSURANCE
CHECKING LINEARITY

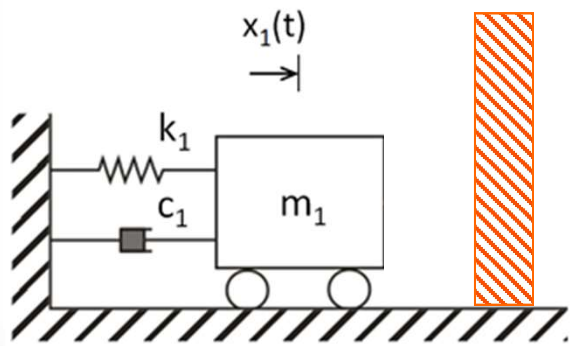
All the theory presented in this course assumes a linear time-invariant system. However, non-linearities are not to be excluded.

Exemplary causes of non-linear behaviour:

- Non-linear material properties e.g. rubbers, plastic
- Geometric constrictions : e.g. hitting a *stopper*
- Geometric deformation (large loads)

Coherence function will indicate presence of non-linearities





QUALITY ASSURANCE

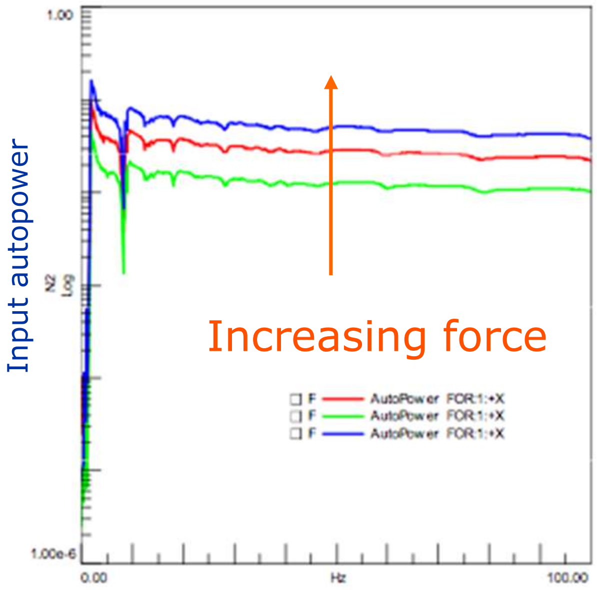
CHECKING LINEARITY

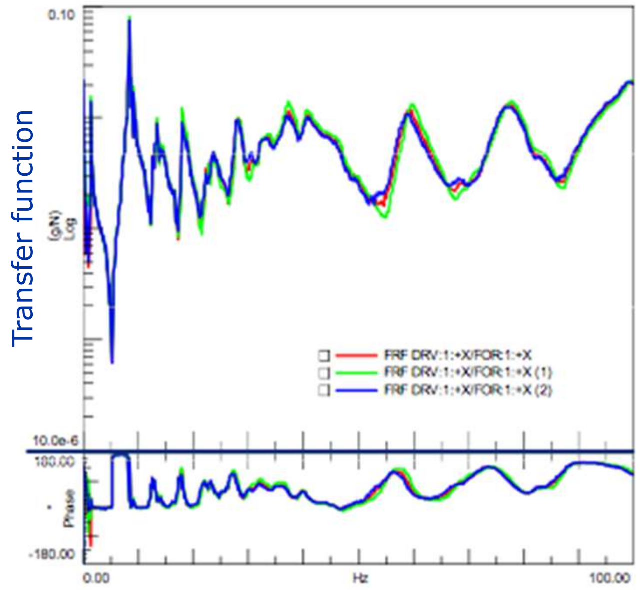
For a linear structure the transfer function is independent of the applied force.

I.e.

$$X(\omega) = H(\omega)F(\omega)$$

$$\alpha X(\omega) = H(\omega)\alpha F(\omega)$$

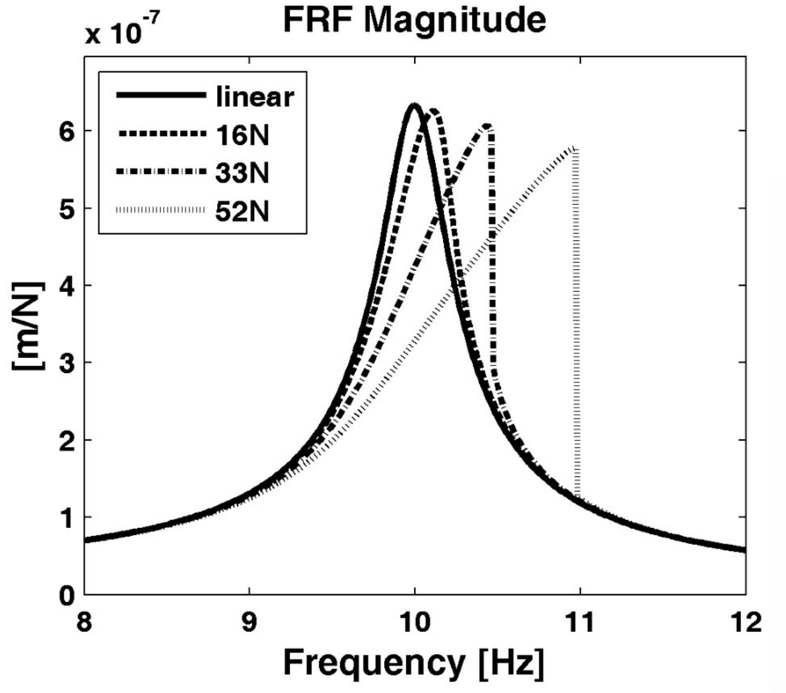




QUALITY ASSURANCE

CHECKING LINEARITY

Textbook example of a non-linear system response under varying loads



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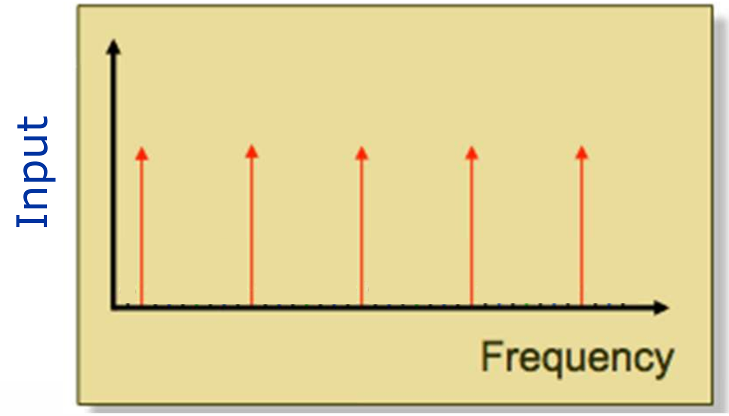
QUALITY ASSURANCE

CHECKING LINEARITY

Advanced strategies are to investigate the ‘off-spectral’ content when using Multi-sine excitation

- Only excite at red frequency lines
  - So not all frequencies that you can excite.
  - A linear system would only respond on these excited frequencies

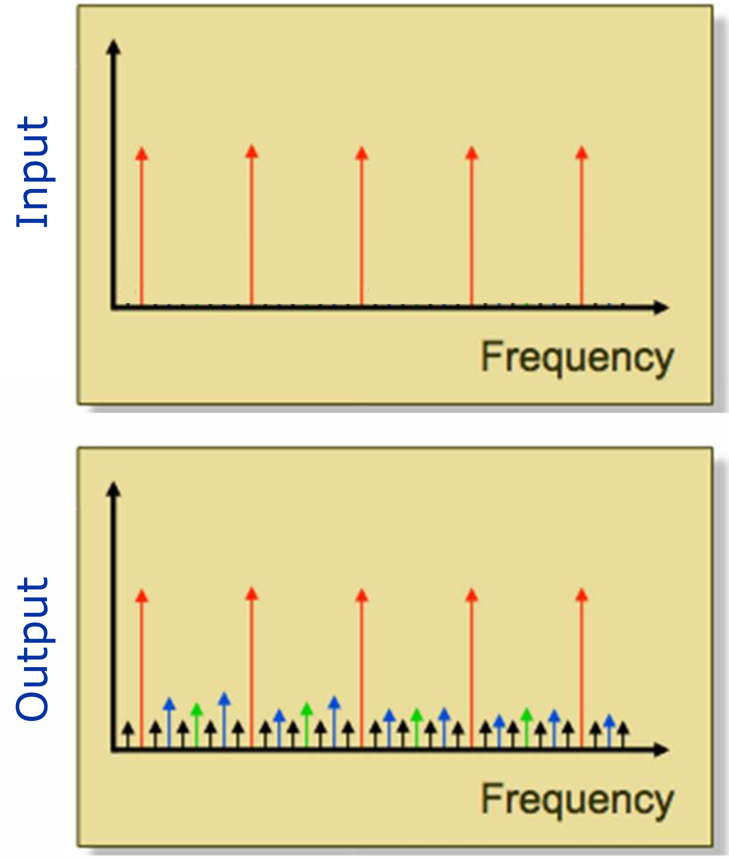


QUALITY ASSURANCE

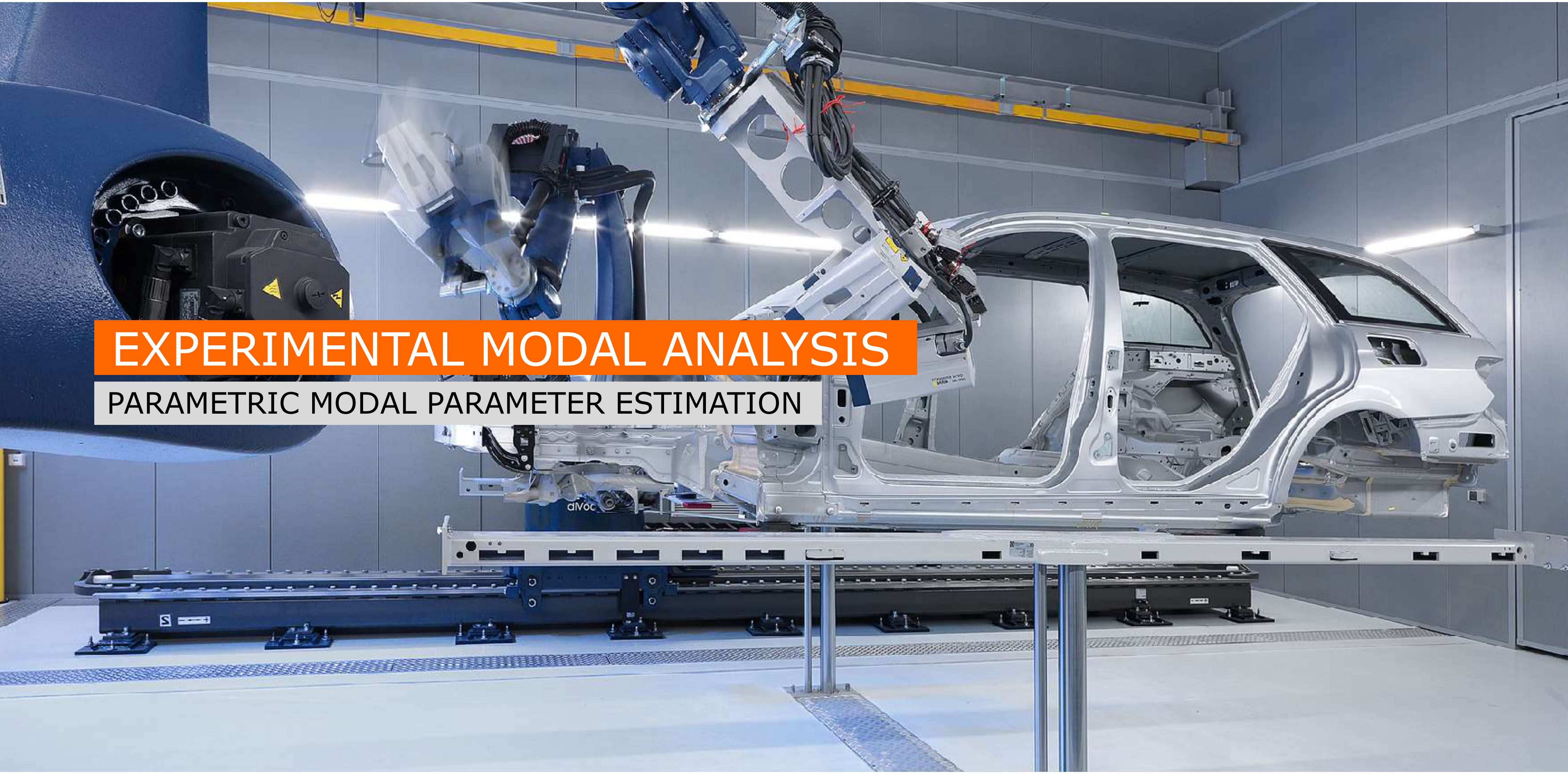
CHECKING LINEARITY

Advanced strategies are to investigate the 'off-spectral' content when using Multi-sine excitation

- Only excite at red frequency lines
  - So not all frequencies that you can excite.
  - A linear system would only respond on these excited frequencies
- Non-linear response manifests at the non-excited frequencies



Read more : Schoukens, J., Rolain, Y., Swevers, J., & De Cuyper, J. (2000). Simple methods and insights to deal with non-linear distortions in FRF-measurements. *Mechanical Systems and Signal Processing*, *14*(4), 657-666.



EXPERIMENTAL MODAL ANALYSIS

PARAMETRIC MODAL PARAMETER ESTIMATION

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EXPERIMENTAL MODAL ANALYSIS

GETTING THE MODAL PARAMETERS FROM MEASUREMENTS

Goal : to determine the three modal parameters from measurements

The three modal parameters are:

EXPERIMENTAL MODAL ANALYSIS

GETTING THE MODAL PARAMETERS FROM MEASUREMENTS

Goal : to determine the three modal parameters from measurements

The three modal parameters are:

- Resonance frequency
- Damping ratio
- Mode shape / Modal participation

EXPERIMENTAL MODAL ANALYSIS

GETTING THE MODAL PARAMETERS FROM MEASUREMENTS

Goal : to determine the three modal parameters from measurements

The three modal parameters are:

- Resonance frequency
- Damping ratio

pole

- Mode shape / Modal participation

Combined they represent the modal model

$$\left[\mathbf{H}(s)\right]_{N_o \times N_i} = \sum_{m=1}^{N_m} \frac{\mathbf{\Psi}_m \mathbf{L}_m^T}{s - \lambda_m} + \frac{\mathbf{\Psi}_m^* \mathbf{L}_m^{*T}}{s - \lambda_m^*}$$

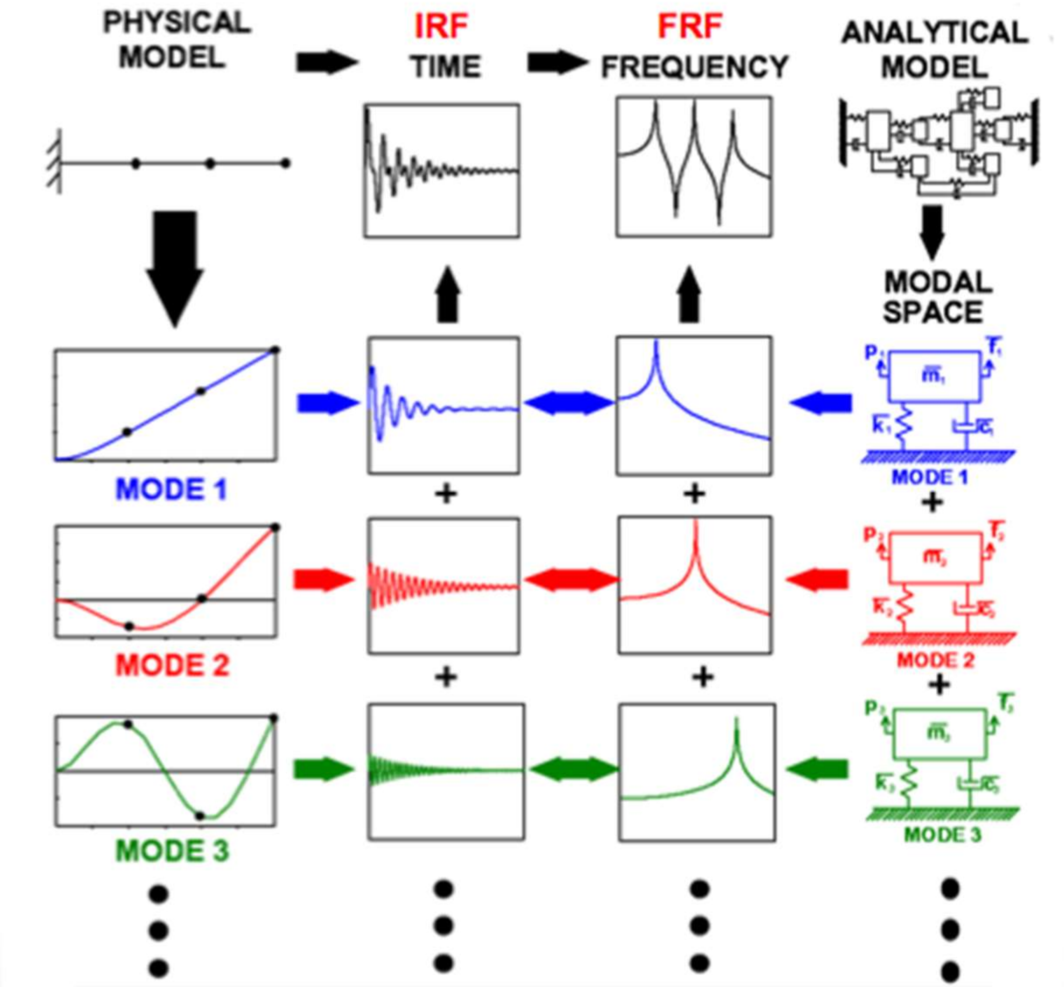
EXPERIMENTAL MODAL ANALYSIS

THE MODAL MODEL

In words:

All dynamics of a MDOF system can be explained as the summed response of SDOF systems

$$[H(s)]_{N_o \times N_i} = \sum_{m=1}^{N_m} \frac{\Psi_m L_m^T}{s - \lambda_m} + \frac{\Psi_m^* L_m^{*T}}{s - \lambda_m^*}$$



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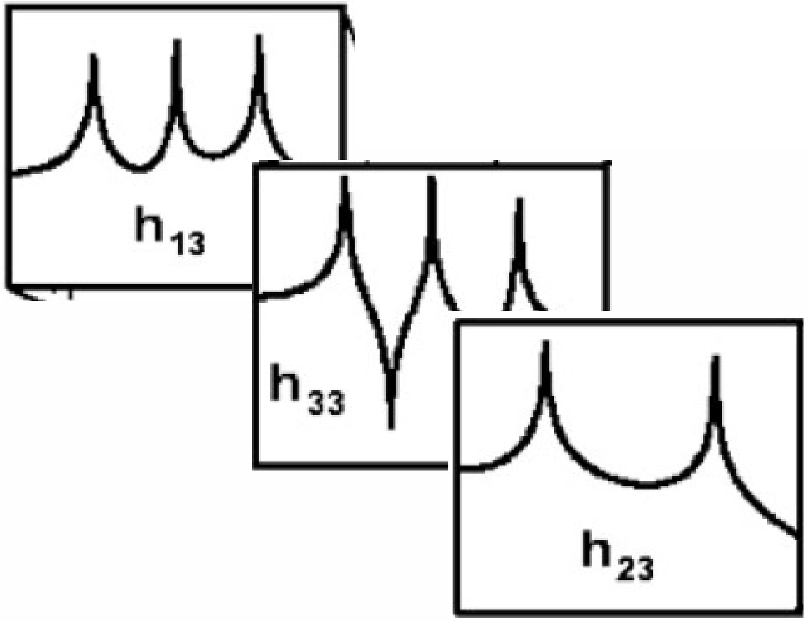
EXPERIMENTAL MODAL ANALYSIS

GETTING THE MODAL PARAMETERS FROM MEASUREMENTS

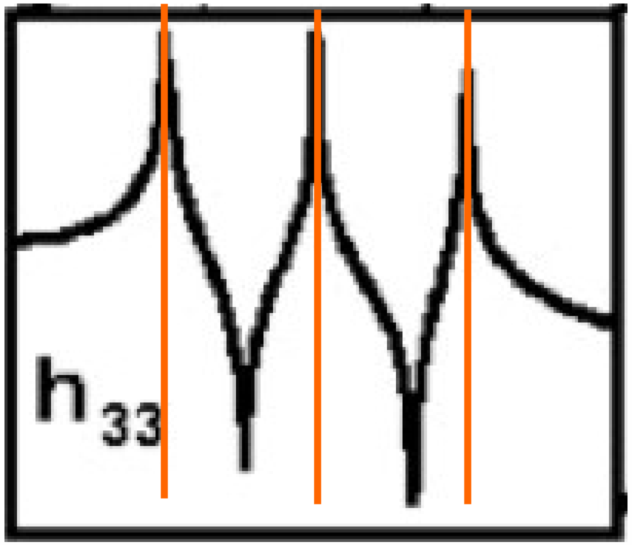


$$[\mathbf{H}(s)]_{N_o \times N_i} = \sum_{m=1}^{N_m} \frac{\boldsymbol{\Psi}_m \mathbf{L}_m^T}{s - \lambda_m} + \frac{\boldsymbol{\Psi}_m^* \mathbf{L}_m^{*T}}{s - \lambda_m^*}$$

GETTING THE MODAL PARAMETERS FROM MEASUREMENTS

Peak picking

Selecting the highest peaks in the FRF and refer to them as resonance frequencies.



Limitations

- Results are limited to the frequency resolution
- Typically only a single FRF at a time (issue with nodal points)
- Does not provide any information on damping

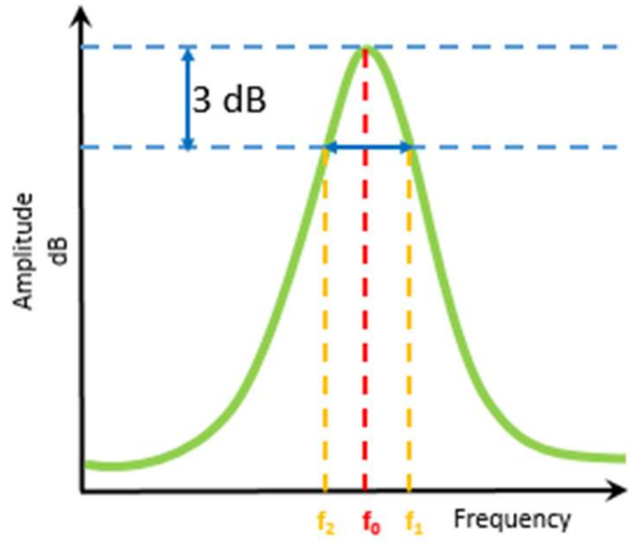
OK for a quick and dirty preliminary assessment… but still used too much as the final result

GETTING THE MODAL PARAMETERS FROM MEASUREMENTS

Peak picking (with damping)

Calculate the damping from the FRF using the 3dB method

Unreliable at best, do not use



$$Q = \frac{f_0}{f_2 - f_1}$$

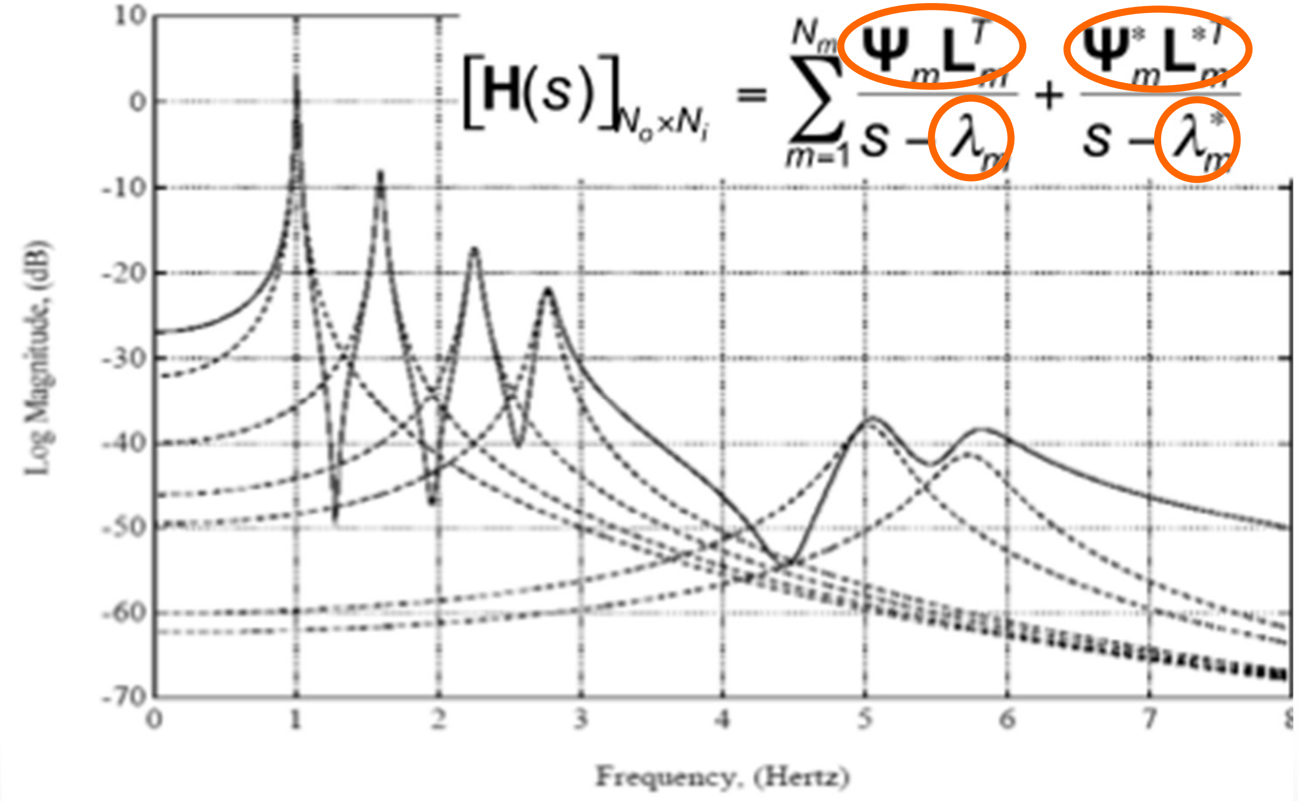
EXPERIMENTAL MODAL ANALYSIS

PREFERRED METHOD

It is preferred to use a curve fitting scheme to estimate the modal parameters from the transfer function measurements.

But the modal model is non linear in the (modal) parameters

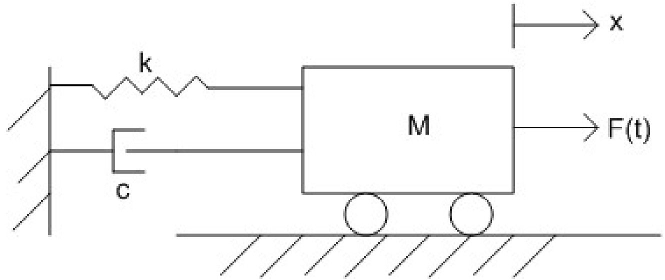
To directly fit the modal model to the FRF would imply a (recursive) non-linear optimisation



$$[\mathbf{H}(s)]_{N_o \times N_i} = \sum_{m=1}^{N_m} \frac{\Psi_m \mathbf{L}_m^T}{s - \lambda_m} + \frac{\Psi_m^* \mathbf{L}_m^{*T}}{s - \lambda_m^*}$$

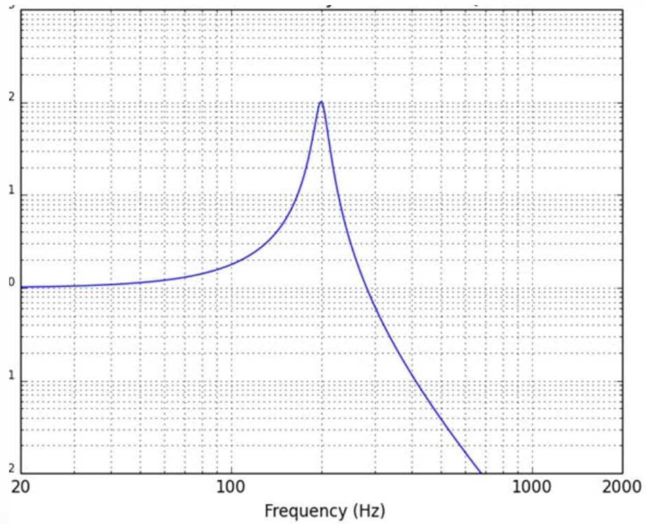
SDOF ESTIMATION

BABY STEPS



Start with a single DOF system

$$X(\omega_i) = \frac{1}{-\omega_i^2 m + j\omega_i c + k} \cdot F(\omega_i)$$



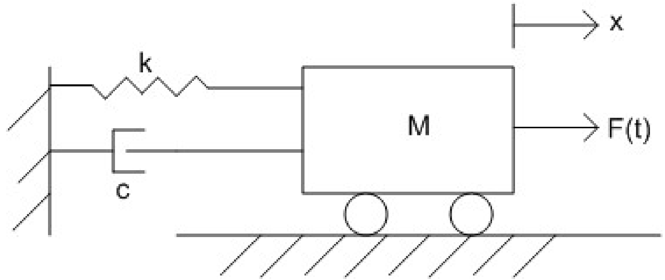
reorganizing the equations

$$\left(-\omega_i^2 m + j\omega_i c + k\right) X(\omega_i) = F(\omega_i)$$

$$\begin{bmatrix} -\omega_i^2 X(\omega_i) & j\omega_i X(\omega_i) & X(\omega_i) \end{bmatrix} \begin{Bmatrix} m \\ c \\ k \end{Bmatrix} = F(\omega_i)$$

SDOF ESTIMATION

BABY STEPS

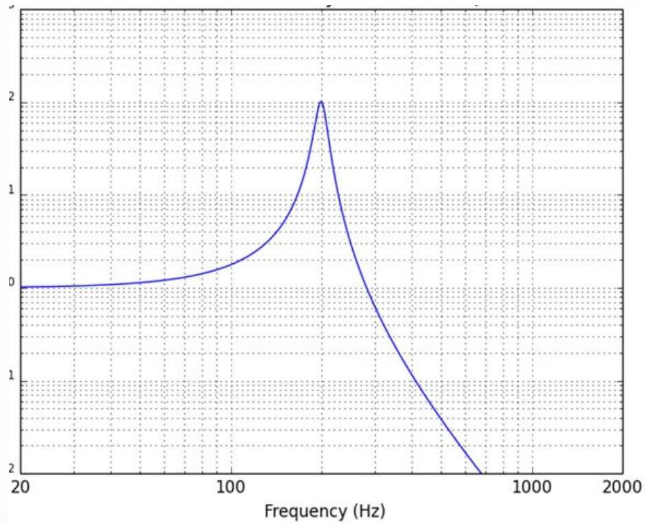


All three unknowns in a single vector

$$\begin{bmatrix} -\omega_i^2\mathbf{X}(\omega_i) & j\omega_i\mathbf{X}(\omega_i) & \mathbf{X}(\omega_i) \end{bmatrix} \boxed{\begin{Bmatrix} \mathbf{m} \\ \mathbf{c} \\ \mathbf{k} \end{Bmatrix}} = \mathbf{F}(\omega_i)$$

This equation holds for all *Nf* measured frequencies

$$\begin{bmatrix} -\omega_1^2\mathbf{X}(\omega_1) & j\omega_1\mathbf{X}(\omega_1) & \mathbf{X}(\omega_1) \\ -\omega_2^2\mathbf{X}(\omega_2) & j\omega_2\mathbf{X}(\omega_2) & \mathbf{X}(\omega_2) \\ & \vdots & \\ -\omega_{N_f}^2\mathbf{X}(\omega_{N_f}) & j\omega_{N_f}\mathbf{X}(\omega_{N_f}) & \mathbf{X}(\omega_{N_f}) \end{bmatrix} \begin{Bmatrix} m \\ c \\ k \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}(\omega_1) \\ \mathbf{F}(\omega_2) \\ \vdots \\ \mathbf{F}(\omega_{N_f}) \end{Bmatrix}$$



SDOF ESTIMATION

BABY STEPS

$$\begin{bmatrix} -\omega_1^2\mathbf{X}(\omega_1) & j\omega_1\mathbf{X}(\omega_1) & \mathbf{X}(\omega_1) \\ -\omega_2^2\mathbf{X}(\omega_2) & j\omega_2\mathbf{X}(\omega_2) & \mathbf{X}(\omega_2) \\ & \vdots & \\ -\omega_{N_f}^2\mathbf{X}(\omega_{N_f}) & j\omega_{N_f}\mathbf{X}(\omega_{N_f}) & \mathbf{X}(\omega_{N_f}) \end{bmatrix} \begin{Bmatrix} m \\ c \\ k \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}(\omega_1) \\ \mathbf{F}(\omega_2) \\ \vdots \\ \mathbf{F}(\omega_{N_f}) \end{Bmatrix}$$

Overdetermined problem

- 3 unknowns **m,c,k** <<< *Nf* equations

Solve for the unknowns θ as to **minimize** the least squares error function LS(θ)

$$K_{(N_f\times 3)}\theta_{(3\times 1)} = \mathbf{F}_{(N_f\times 1)}$$

$$\mathbf{e}_{(N_f\times 1)} = K_{(N_f\times 3)}\theta_{(3\times 1)} - \mathbf{F}_{(N_f\times 1)}$$

$$LS(\theta) = \mathbf{e}^T\mathbf{e}$$

$$\frac{\delta LS}{\delta\theta} = 0$$

SDOF ESTIMATION

BABY STEPS

The values of theta found by solving the bottom equation represent the least squares estimate of the problem posed,

They are the set of parameters that best fit with the data contained in the measurements

We found **m,c,k**

$$\mathbf{e}_{(N_f \times 1)} = K_{(N_f \times 3)} \theta_{(3 \times 1)} - \mathbf{F}_{(N_f \times 1)}$$

$$LS(\theta) = \mathbf{e}^T \mathbf{e}$$

$$\frac{\delta LS}{\delta \theta} = 0$$

$$2 \frac{\delta \mathbf{e}^T}{\delta \theta} \mathbf{e} = 0$$

$$2K^T \cdot (K \theta_{\mathbf{LS}} - \mathbf{F}) = 0$$

$$\boxed{\begin{aligned} \theta_{\mathbf{LS}} &= \left(K^T K\right)^{-1} K^T \mathbf{F} \\ &= K \setminus \mathbf{F} \end{aligned}}$$

SDOF ESTIMATION

BABY STEPS

We found **m,c,k** how to get the modal parameters?

$$X(\omega_i) = \frac{1}{-\omega_i^2 m + j\omega_i c + k} \cdot F(\omega_i)$$

The system poles (λ) are the roots of the denominator, so just use the found *m,c,k* values to calculate the roots.

E.g. in MATLAB using the `roots` command

From the poles the resonance frequency and damping ratios are calculated

$$f_{res} = \frac{abs(\lambda)}{2\pi}$$

$$\xi = \frac{-real(\lambda)}{||\lambda||}$$

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SDOF ESTIMATION

ADDING A FINAL CONSTRAINT

In fact, the results of the previous approach will be two values that solve the equation

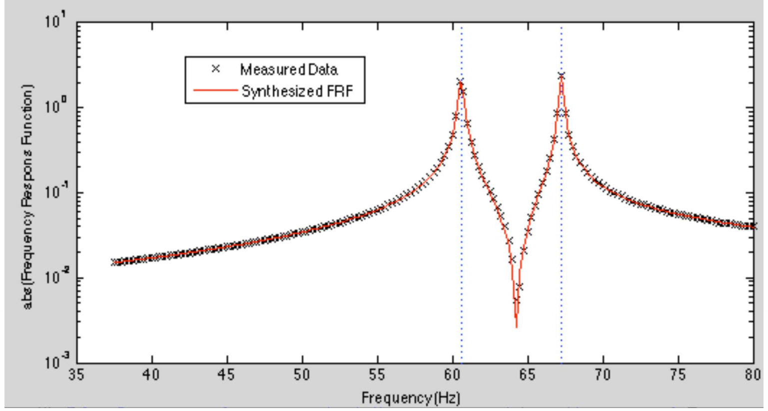
$mx^2+cx+k=0$

There is no guarantee that these two results are eachothers complex conjugate. An additional constraint is necessary to force this on the least squares solution.

$$\begin{bmatrix} real(K) \\ imag(K) \end{bmatrix} \theta = \begin{Bmatrix} real(\mathbf{F}) \\ imag(\mathbf{F}) \end{Bmatrix}$$

EXTENDING TO MULTIPLE MODES

ADDING ZEROS



Two things change when more modes are considered:

- The model order has to be increased to allow for a larger number of modes
- Zeros need to be considered

Previous single degree of freedom model is expanded to a MDOF nominator-denominator model in which the model orders (*Nn*, *Nd*) exceed 2

$$H(\omega_i) = \frac{\sum_{l=0}^{N_n} n_l (j\omega_i)^l}{\sum_{k=0}^{N_d} d_k (j\omega_i)^k}$$

EXTENDING TO MULTIPLE MODES

ADDING ZEROS

$$H(\omega_i) = \frac{\sum_{l=0}^{N_n} n_l (j\omega_i)^l}{\sum_{k=0}^{N_d} d_k (j\omega_i)^k}$$

Solve for the unknowns n_l, and d_k

$$\left(\sum_{k=0}^{N_d} d_k (j\omega_i)^k\right) H(\omega_i) - \sum_{l=0}^{N_n} n_l (j\omega_i)^l = 0$$

Rewrite equations to something of the form

$$K\theta = 0$$

Volunteers?

EXTENDING TO MULTIPLE MODES

ADDING ZEROS

$$\left(\sum_{k=0}^{N_d} d_k (j\omega_i)^k\right) H(\omega_i) - \sum_{l=0}^{N_n} n_l (j\omega_i)^l = 0$$

$$\begin{bmatrix} H(\omega_1) & j\omega_1 H(\omega_1) & (j\omega_1)^2 H(\omega_1) & \dots & (j\omega_1)^{N_d} H(\omega_1) & -1 & -j\omega_1 & -(j\omega_1)^2 & \dots & -(j\omega_1)^{N_n} \\ H(\omega_2) & j\omega_2 H(\omega_2) & (j\omega_2)^2 H(\omega_2) & \dots & (j\omega_2)^{N_d} H(\omega_2) & -1 & -j\omega_2 & -(j\omega_2)^2 & \dots & -(j\omega_2)^{N_n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ H(\omega_{N_f}) & j\omega_{N_f} H(\omega_{N_f}) & (j\omega_{N_f})^2 H(\omega_{N_f}) & \dots & (j\omega_{N_f})^{N_d} H(\omega_{N_f}) & -1 & -j\omega_{N_f} & -(j\omega_{N_f})^2 & \dots & -(j\omega_{N_f})^{N_n} \end{bmatrix} \left\{ \begin{matrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N_d} \\ n_0 \\ n_1 \\ n_2 \\ \vdots \\ n_{N_n} \end{matrix} \right\} = 0$$

$$K\theta = 0$$

EXTENDING TO MULTIPLE MODES

APPLYING CONSTRAINT

Next it remains to solve $$K\theta = 0$$

But, an infinite number of solutions exist (i.e. any multiple of θ) !

To solve for theta the number of solutions has to be constrained to a single least squares solution.

Possible constraints are :

- $d_{nd} = 1$: Most common assumption
- $|\theta| = 1$: Total least squares solution

EXTENDING TO MULTIPLE MODES

APPLYING CONSTRAINT

$$\begin{bmatrix} H(\omega_1) & j\omega_1 H(\omega_1) & (j\omega_1)^2 H(\omega_1) & \dots & (j\omega_1)^{N_d} H(\omega_1) & -1 & -j\omega_1 & -(j\omega_1)^2 & \dots & -(j\omega_1)^{N_n} \\ H(\omega_2) & j\omega_2 H(\omega_2) & (j\omega_2)^2 H(\omega_2) & \dots & (j\omega_2)^{N_d} H(\omega_2) & -1 & -j\omega_2 & -(j\omega_2)^2 & \dots & -(j\omega_2)^{N_n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ H(\omega_{N_f}) & j\omega_{N_f} H(\omega_{N_f}) & (j\omega_{N_f})^2 H(\omega_{N_f}) & \dots & (j\omega_{N_f})^{N_d} H(\omega_{N_f}) & -1 & -j\omega_{N_f} & -(j\omega_{N_f})^2 & \dots & -(j\omega_{N_f})^{N_n} \end{bmatrix} \begin{Bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ \mathbf{1} \\ n_0 \\ n_1 \\ n_2 \\ \vdots \\ n_{N_n} \end{Bmatrix} = 0$$

$$\begin{bmatrix} H(\omega_1) & j\omega_1 H(\omega_1) & (j\omega_1)^2 H(\omega_1) & \dots -1 & -j\omega_1 & -(j\omega_1)^2 & \dots & -(j\omega_1)^{N_n} \\ H(\omega_2) & j\omega_2 H(\omega_2) & (j\omega_2)^2 H(\omega_2) & \dots -1 & -j\omega_2 & -(j\omega_2)^2 & \dots & -(j\omega_2)^{N_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ H(\omega_{N_f}) & j\omega_{N_f} H(\omega_{N_f}) & (j\omega_{N_f})^2 H(\omega_{N_f}) & \dots -1 & -j\omega_{N_f} & -(j\omega_{N_f})^2 & \dots & -(j\omega_{N_f})^{N_n} \end{bmatrix} \begin{Bmatrix} d_o \\ d_1 \\ d_2 \\ \vdots \\ n_0 \\ n_1 \\ n_2 \\ \vdots \\ n_{n_n} \end{Bmatrix} = - \begin{Bmatrix} (j\omega_1)^{n_d} H(\omega_1) \\ (j\omega_2)^{n_d} H(\omega_2) \\ \vdots \\ (j\omega_f)^{n_d} H(\omega_f) \end{Bmatrix}$$

And we can solve again using Theta=K\Y

EXTENDING TO MULTIPLE MODES

SOLVING FOR THE MODAL PARAMETERS

Again the poles are found as the roots of the denominator polynomial :

$$\sum_{k=0}^{N_d} d_k (j\omega_i)^k = 0$$

This can be done e.g. using the `roots` command in MATLAB

EXTENDING TO MULTIPLE MODES

MODEL ORDERS

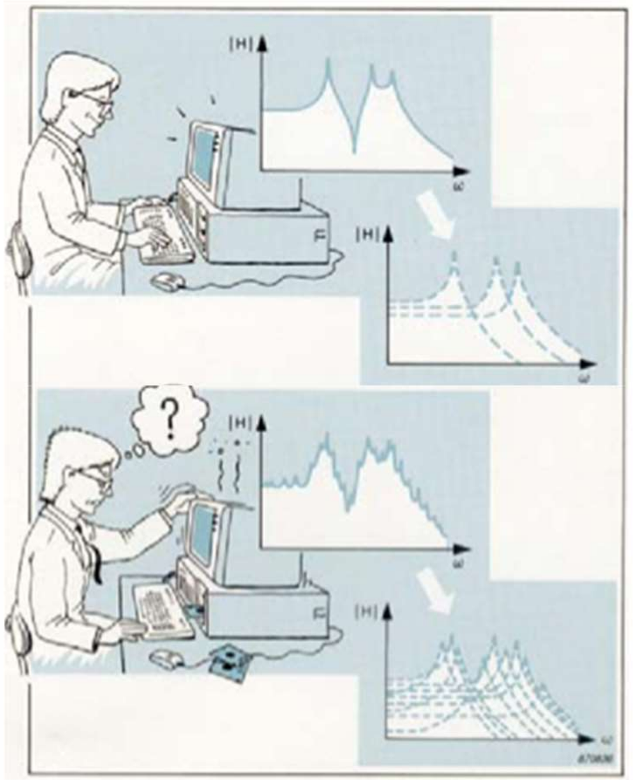
The model orders (N_n and N_d) are settings by the user. There is really no upper bound,…

With increasing model orders the model fit will get better and the LS error will reduce….

But a lot of numerical modes (without physical meaning) will appear as the model cannot distinguish between physical modes and noise.

Always keep in mind that the curve fitter only aims to minimize the LS cost function, not to improve the "physicality" of the result.

There is thus a tradeof for higher model orders.



STABILIZATION DIAGRAM

A TOOL TO DETERMINE STRUCTURAL MODES

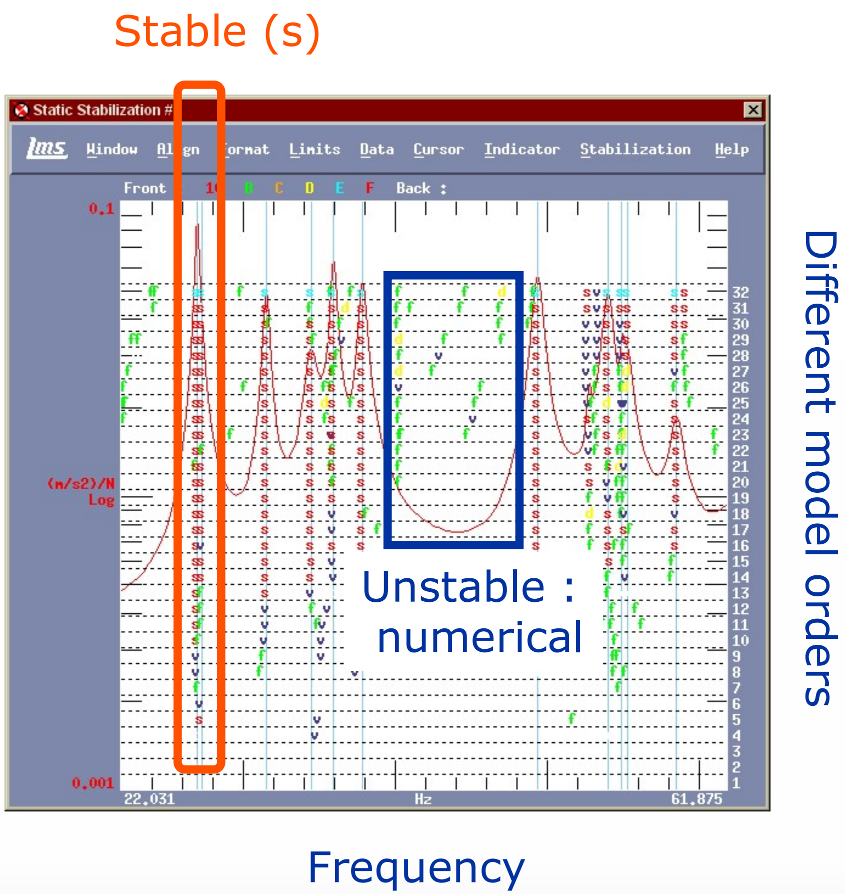
Typically a so called stabilization chart is used to distinguish between structural and numerical modes.

Basic assumption :

Structural models will appear in models of different model order while numerical modes will vary for different model orders.

Thus by calculating the poles for different model orders , structural poles will appear in all model orders. They are "stable" (s).

Rule of thumb : choose the lowest model order where all modes of interest are stable



NUMERICAL UNBALANCE

USING ORTHOGONAL BASE FUNCTIONS

One of the common issues when implementing the frequency domain estimators is a numerical unbalance at high model orders.

$$
\begin{matrix}
-1 & -j\omega_1 & -(j\omega_1)^2 & \dots & -(j\omega_1)^{N_n} \\
-1 & -j\omega_2 & -(j\omega_2)^2 & \dots & -(j\omega_2)^{N_n} \\
\vdots & \vdots & \vdots & & \vdots \\
-1 & -j\omega_{N_f} & -(j\omega_{N_f})^2 & \dots & -(j\omega_{N_f})^{N_n}
\end{matrix}
$$

For high model orders the values in the first column and the last column differ in several orders of magnitude

e.g. for model order 64, and the row with ω=10

- First column : -1
- Last column : 10^64

This unbalance leeds to numerical issues and ultimately erroneous results (e.g. a poor fit at low frequencies).

NUMERICAL UNBALANCE

USING ORTHOGONAL BASE FUNCTIONS

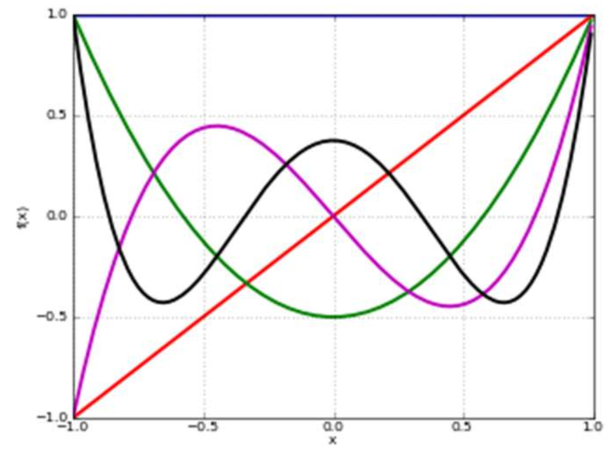
To avoid these issues a first step is to normalize the frequency band (e.g. map all frequencies from 0 to 1).

But even better is the use of orthogonal base functions, which improve the conditioning of the problem altogether :

$$H(\omega_i) = \frac{\sum_{l=0}^{N_n} n_l \Omega_l(\omega_i)}{\sum_{k=0}^{N_d} d_k \Omega_k(\omega_i)}$$

$$\Omega_j(\omega_f) = \begin{cases} (i\omega_f)^j \\ \exp(-i\omega_f \cdot j) \\ \text{Forsythe polynomials, ...} \end{cases}$$

recommended



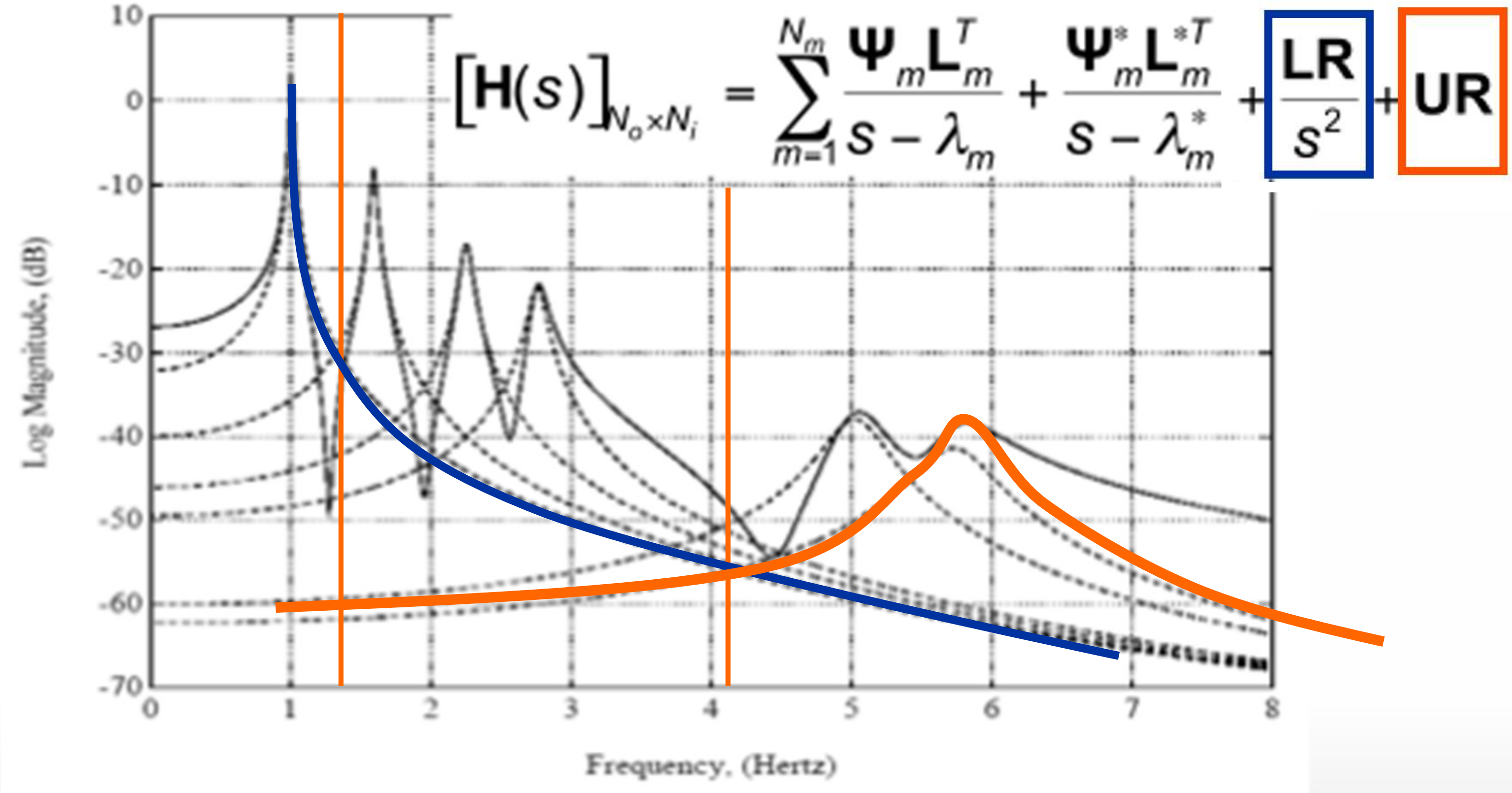
Note that if another polynomial is used, the found roots of the denominator have to be mapped back the frequency domain

UPPER AND LOWER RESIDUALS

AN ADDITION WHEN PROCESSING A LIMITED BANDWIDTH

In general only a limited bandwidth is curve fitted. This implies some modes are out of this frequency band. Nonetheless these modes have an influence in the band of interest.

These residual terms can be added to the numerator nominator model as two additional unknowns



$$[\mathbf{H}(s)]_{N_o\times N_i} = \sum_{m=1}^{N_m} \frac{\Psi_m \mathbf{L}_m^T}{s-\lambda_m} + \frac{\Psi_m^* \mathbf{L}_m^{*T}}{s-\lambda_m^*} + \boxed{\frac{\mathbf{LR}}{s^2}} + \boxed{\mathbf{UR}}$$

FROM SDOF TO MDOF

PROCESSING THE WHOLE FRF

Up to now we were not able to determine the mode shapes. To do so the curve fitting algorithm has to be extended to process the entire FRF.

$$H_{11}(\omega) = \frac{\sum_{l=0}^{N_n} n_{11,l}\Omega_l(\omega)}{\sum_{k=0}^{N_d} d_k\Omega_k(\omega)} \quad H_{12}(\omega) = \frac{\sum_{l=0}^{N_n} n_{12,l}\Omega_l(\omega)}{\sum_{k=0}^{N_d} d_k\Omega_k(\omega)} \qquad H_{ij}(\omega) = \frac{\sum_{l=0}^{N_n} n_{ij,l}\Omega_l(\omega)}{\sum_{k=0}^{N_d} d_k\Omega_k(\omega)}$$

Consistent with the modal model, poles are global, the denominator is common to all equations.

-> **Common denominator model!**

FROM SDOF TO MDOF

BUILDING THE OBSERVATION MATRIX

$$
\begin{pmatrix}
\begin{bmatrix} -1 & -j\omega_1 & -(j\omega_1)^2 & \dots & -(j\omega_1)^{N_n} \\ -1 & -j\omega_2 & -(j\omega_2)^2 & \dots & -(j\omega_2)^{N_n} \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -j\omega_{N_f} & -(j\omega_{N_f})^2 & \dots & -(j\omega_{N_f})^{N_n} \end{bmatrix} & & & \underbrace{\begin{bmatrix} H(\omega_1) & j\omega_1 H(\omega_1) & (j\omega_1)^2 H(\omega_1) & \dots & (j\omega_1)^{N_d} H(\omega_1) \\ H(\omega_2) & j\omega_2 H(\omega_2) & (j\omega_2)^2 H(\omega_2) & \dots & (j\omega_2)^{N_d} H(\omega_2) \\ \vdots & \vdots & \vdots & & \vdots \\ H(\omega_{N_f}) & j\omega_{N_f} H(\omega_{N_f}) & (j\omega_{N_f})^2 H(\omega_{N_f}) & \dots & (j\omega_{N_f})^{N_d} H(\omega_{N_f}) \end{bmatrix}}_{H11} \\
& \begin{bmatrix} -1 & -j\omega_1 & -(j\omega_1)^2 & \dots & -(j\omega_1)^{N_n} \\ -1 & -j\omega_2 & -(j\omega_2)^2 & \dots & -(j\omega_2)^{N_n} \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -j\omega_{N_f} & -(j\omega_{N_f})^2 & \dots & -(j\omega_{N_f})^{N_n} \end{bmatrix} & & \underbrace{\begin{bmatrix} H(\omega_1) & j\omega_1 H(\omega_1) & (j\omega_1)^2 H(\omega_1) & \dots & (j\omega_1)^{N_d} H(\omega_1) \\ H(\omega_2) & j\omega_2 H(\omega_2) & (j\omega_2)^2 H(\omega_2) & \dots & (j\omega_2)^{N_d} H(\omega_2) \\ \vdots & \vdots & \vdots & & \vdots \\ H(\omega_{N_f}) & j\omega_{N_f} H(\omega_{N_f}) & (j\omega_{N_f})^2 H(\omega_{N_f}) & \dots & (j\omega_{N_f})^{N_d} H(\omega_{N_f}) \end{bmatrix}}_{H12} \\
& & \begin{bmatrix} -1 & -j\omega_1 & -(j\omega_1)^2 & \dots & -(j\omega_1)^{N_n} \\ -1 & -j\omega_2 & -(j\omega_2)^2 & \dots & -(j\omega_2)^{N_n} \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -j\omega_{N_f} & -(j\omega_{N_f})^2 & \dots & -(j\omega_{N_f})^{N_n} \end{bmatrix} & \underbrace{\begin{bmatrix} H(\omega_1) & j\omega_1 H(\omega_1) & (j\omega_1)^2 H(\omega_1) & \dots & (j\omega_1)^{N_d} H(\omega_1) \\ H(\omega_2) & j\omega_2 H(\omega_2) & (j\omega_2)^2 H(\omega_2) & \dots & (j\omega_2)^{N_d} H(\omega_2) \\ \vdots & \vdots & \vdots & & \vdots \\ H(\omega_{N_f}) & j\omega_{N_f} H(\omega_{N_f}) & (j\omega_{N_f})^2 H(\omega_{N_f}) & \dots & (j\omega_{N_f})^{N_d} H(\omega_{N_f}) \end{bmatrix}}_{H13}
\end{pmatrix}
\begin{Bmatrix}
\underbrace{\begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ \vdots \\ n_{n_n} \end{bmatrix}}_{n11} \\
\underbrace{\begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ \vdots \\ n_{n_n} \end{bmatrix}}_{n12} \\
\underbrace{\begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ \vdots \\ n_{n_n} \end{bmatrix}}_{n13} \\
\begin{bmatrix} d_o \\ d_1 \\ d_2 \\ \vdots \\ d_{n_d} \end{bmatrix}
\end{Bmatrix}
$$

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FROM SDOF TO MDOF

SOLVING FOR THE MODAL PARAMETERS

Again we obtain the model coefficients of both the denominator and the numerator.

As ever the poles are found by solving for the roots of the (common) denominator

$$\sum_{k=0}^{N_d} d_k (j\omega_i)^k = 0$$

This can be done e.g. using the `roots` command in MATLAB

$$[\mathbf{H}(s)]_{N_o \times N_i} = \sum_{m=1}^{N_m} \frac{\mathbf{\Psi}_m \mathbf{L}_m^T}{s - \lambda_m} + \frac{\mathbf{\Psi}_m^* \mathbf{L}_m^{*T}}{s - \lambda_m^*}$$

FROM SDOF TO MDOF

SOLVING FOR THE MODE SHAPES

The mode shapes are enclosed in the numerator polynomial. We need to evaluate in the found system poles lambda

$$H(\omega_m)(j\omega_m - \lambda_m) = A_m$$

These residuals Am are related to the mode shape and modal participation vectors

$$A_m \sim \Psi_m L_m^T$$

In theory Am would be of Rank 1, but in practice it will not : Singular value decomposition (SVD)

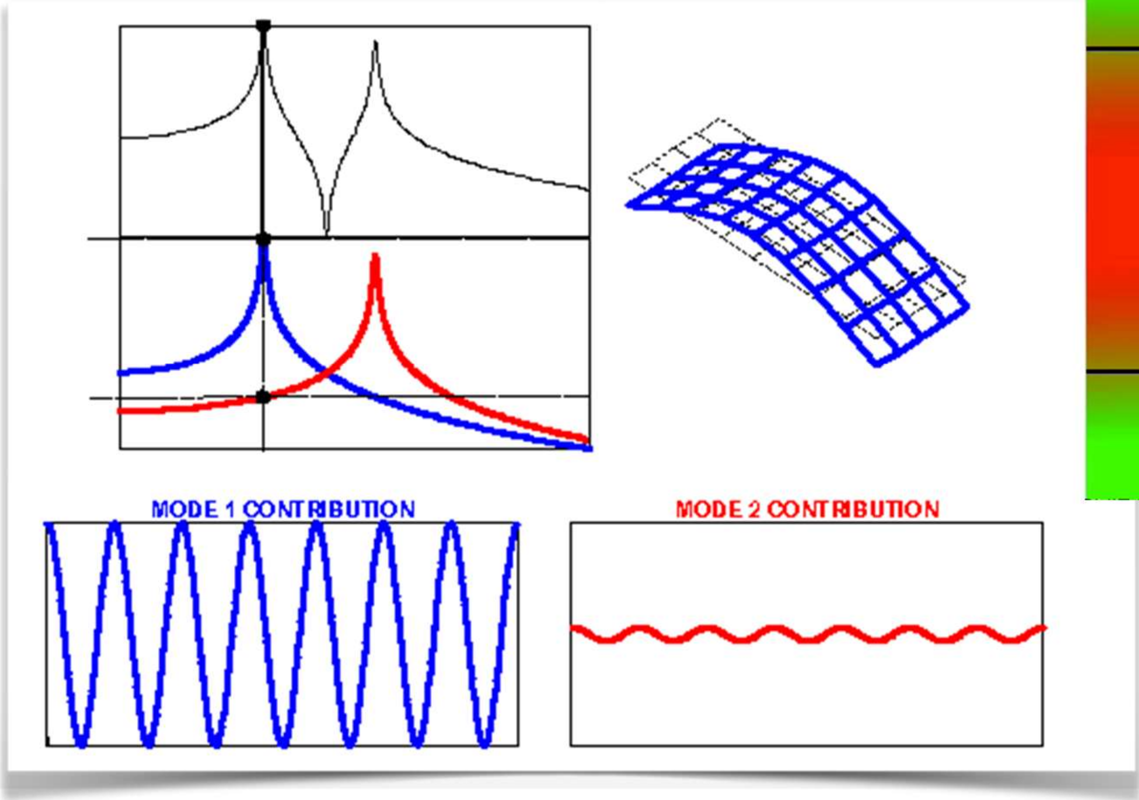
$$A_m = U \Sigma V^T$$

The mode shape Psi_m will be as the first column of U, the first column of V is the modal participation L_m

FROM SDOF TO MDOF

The difference between ODS and modeshapes

Modes are orthogonal



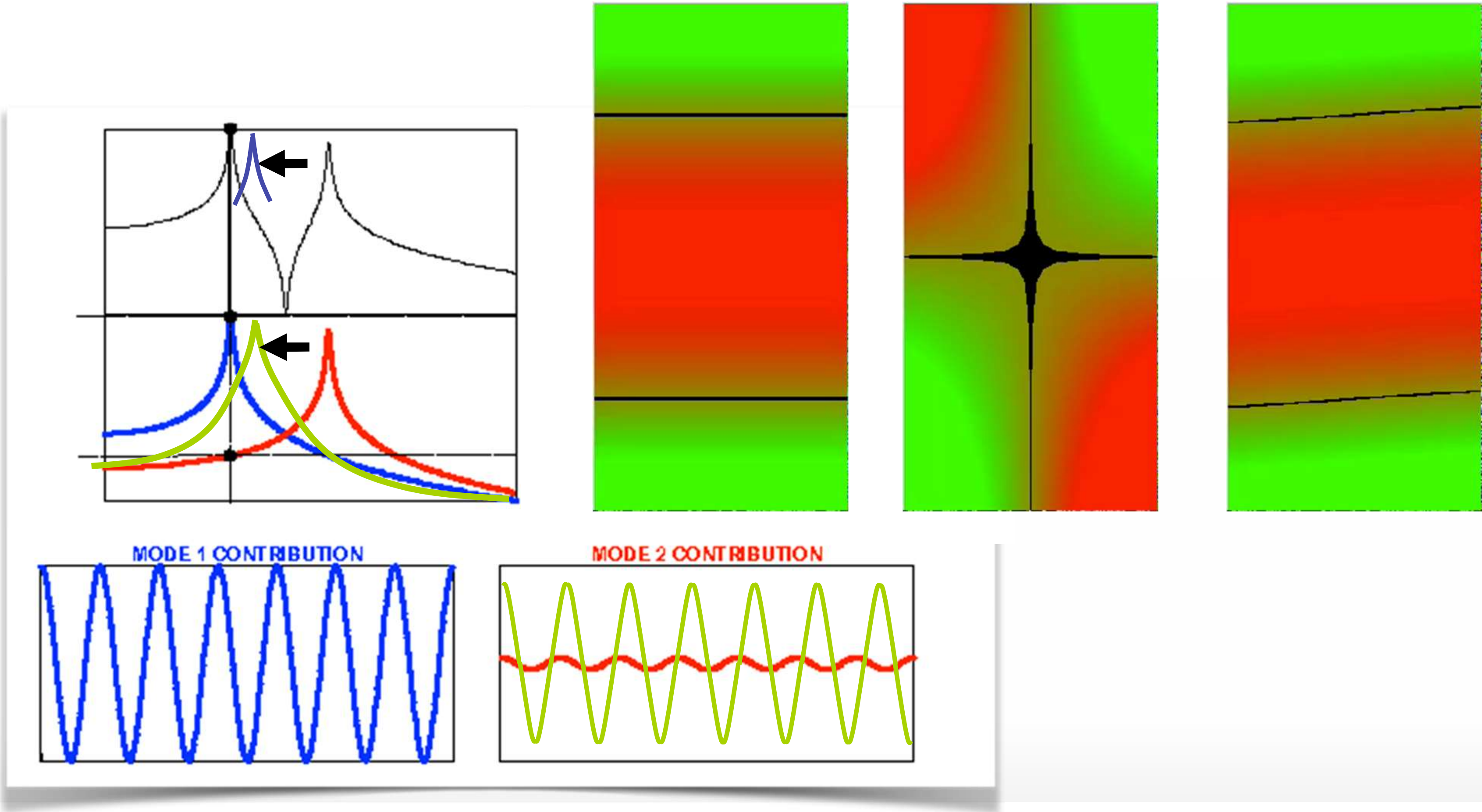
FROM SDOF TO MDOF

The difference between ODS and modeshapes

Modes are orthogonal

ODS are a linear combination of modes

$$ODS = \alpha \Psi_i + \beta \Psi_j$$



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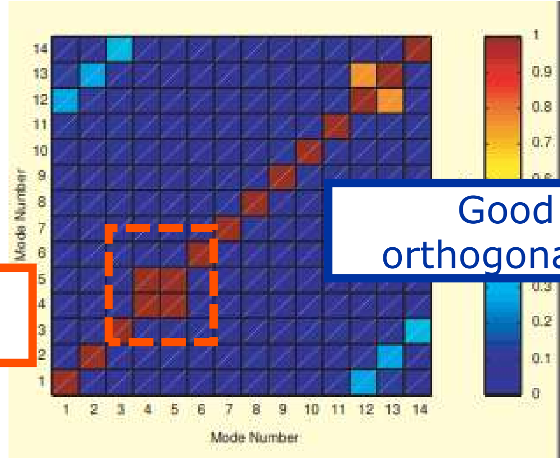
MODAL ASSURANCE CRITERIUM

A WAY TO CHECK YOUR RESULTS



Check the orthogonality of the found modeshapes

$$MAC(\Psi_i, \Psi_j) = \frac{\left\| \Psi_i^T \Psi_j^* \right\|^2}{(\Psi_i^T \Psi_i^*)(\Psi_j^T \Psi_j^*)}$$

MAC ranges from 1 to 0 with MAC(i,i) =1. And perfect orthogonality = 0

ODS result in MAC areas of higher MAC as ODS are the super positions of Mode shapes

$$MAC(ODS, \Psi_i) \sim \alpha \qquad\qquad MAC(ODS, \Psi_j) \sim \beta$$

OTHER APPROACHES

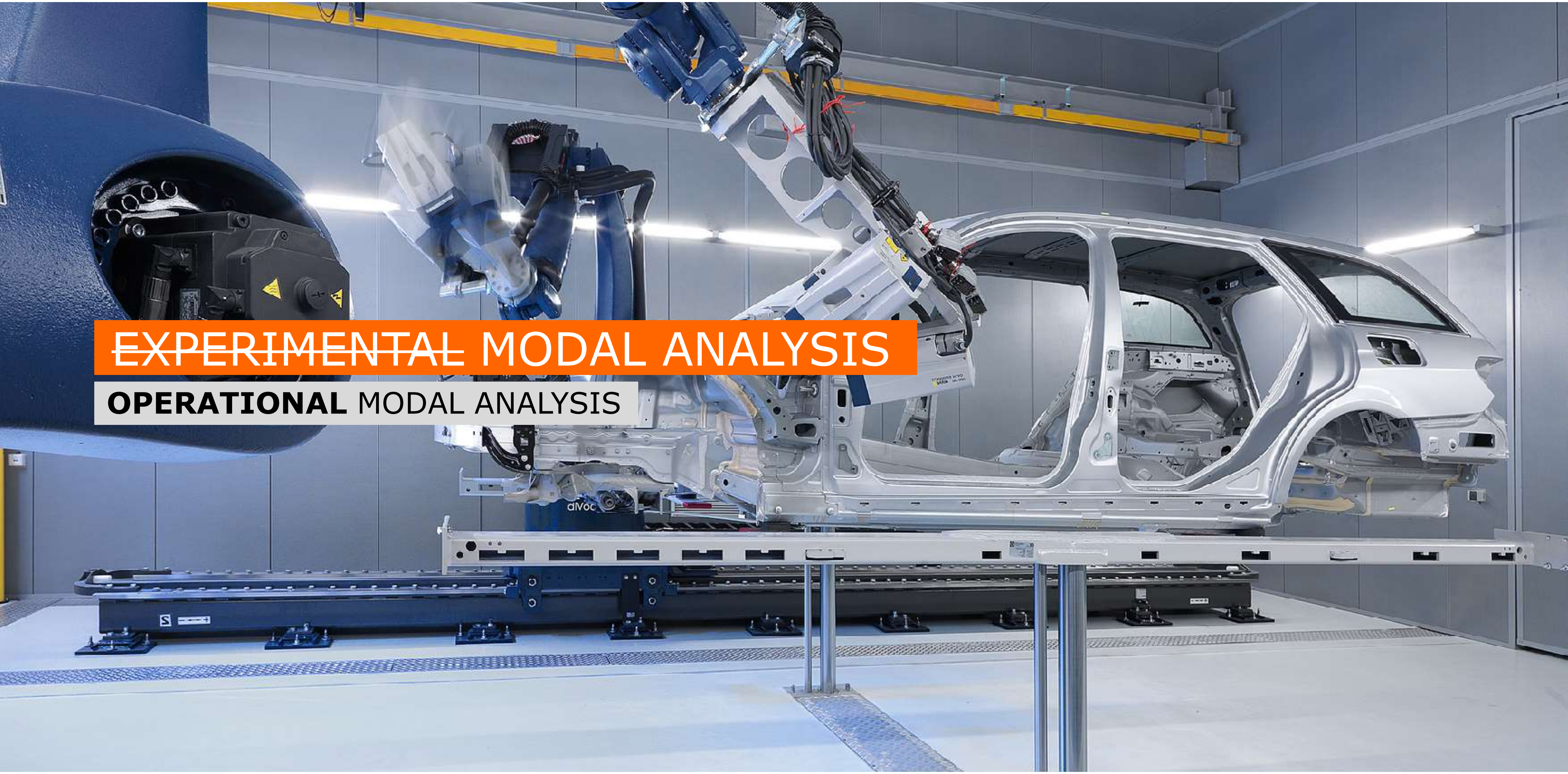
THERE IS MORE THAN ONE WAY TO SOLVE FOR THE MODAL PARAMETERS

In this lesson we discussed the use of a least squares frequency domain estimator (LSCF)

But other solutions exist:

- In time domain, e.g. LSCE uses decaying exponentials as base function
- pLSCF (Polymax) uses a matrix polynomial as denominator
- Solutions based on the state space model
- Weighted least squares or Maximum likelihood cost functions instead of the herein used LS approach

$$V_{WLS}(\boldsymbol{\theta}) = \sum_{f=1}^{N_f} \sum_{k=1}^{N_o N_i} \frac{\left| d(\boldsymbol{\theta},\omega_f) H_k(\omega_f) - N_k(\boldsymbol{\theta},\omega_f) \right|^2}{w_k(\omega_f)} \qquad V_{MLE}(\boldsymbol{\theta}) = \sum_{f=1}^{N_f} \sum_{k=1}^{N_o N_i} \frac{\left| d(\boldsymbol{\theta},\omega_f) H_k(\omega_f) - N_k(\boldsymbol{\theta},\omega_f) \right|^2}{\left| d(\boldsymbol{\theta},\omega_f) \right|^2 \operatorname{var}\{H_k(\omega_f)\}}$$



~~EXPERIMENTAL~~ MODAL ANALYSIS

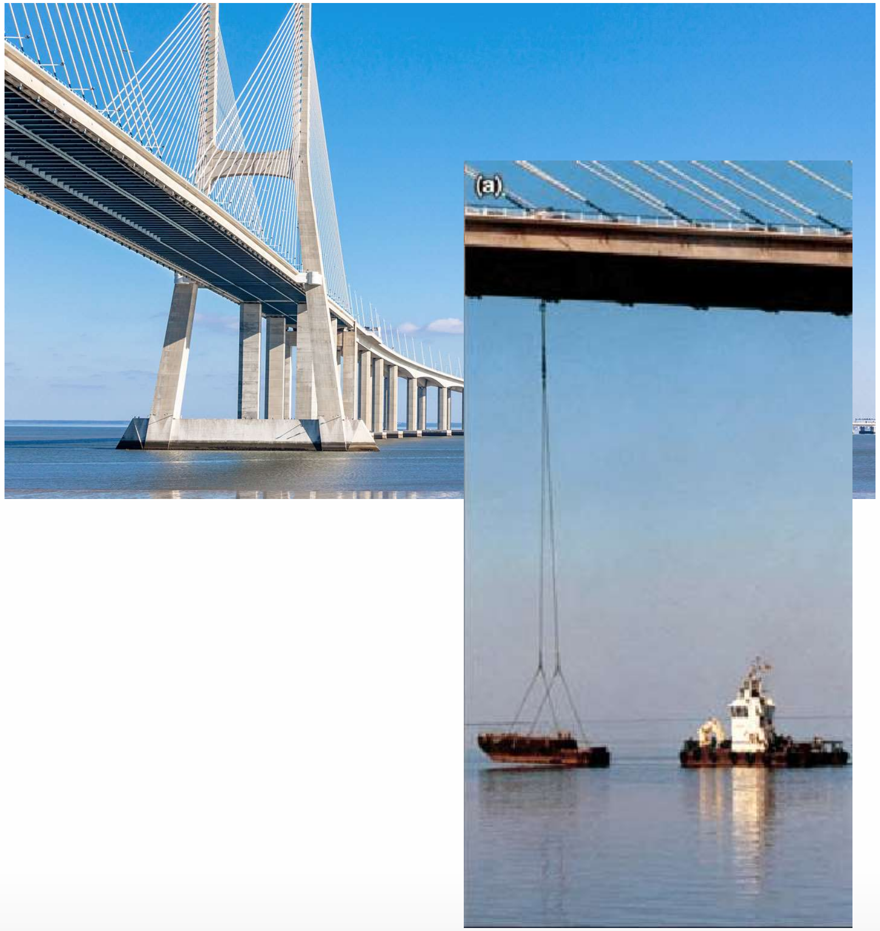
OPERATIONAL MODAL ANALYSIS

OPERATIONAL MODAL ANALYSIS

WHAT IF THE LOAD IS UNKNOWN?

For very large structures it is hard to apply controlled forces.

Example : Vasco da Gama bridge, Lisbon Portugal



OPERATIONAL MODAL ANALYSIS

WHAT IF THE LOAD IS UNKNOWN?

Operational modal analysis uses the natural, but unknown, excitation instead of an applied force.

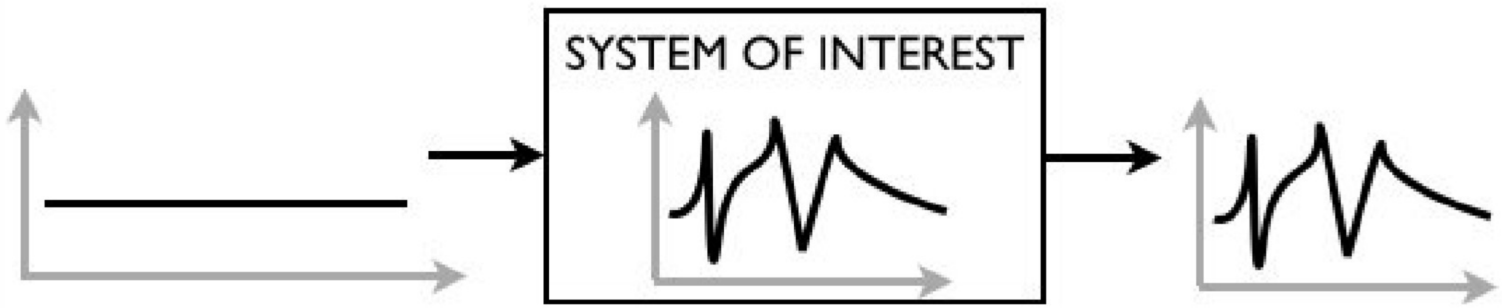
Applications :

- Large structures
- Structural health monitoring: i.e. continuous Modal Analysis
- Flutter testing : Actual boundary condition, actual loads

OPERATIONAL MODAL ANALYSIS

WHAT IF THE LOAD IS UNKNOWN?

The basic assumption of operational modal analysis (OMA) is that the spectrum of the inputs is white noise*.



** Of course we also assume the system of interest is linear and time-invariant*

OPERATIONAL MODAL ANALYSIS

EXAMPLE ON OMA

Continuous operational modal analysis performed on the Infante D. Henrique bridge in Lisbon, Portugal.

Setup consisting of accelerometers and temperature sensors

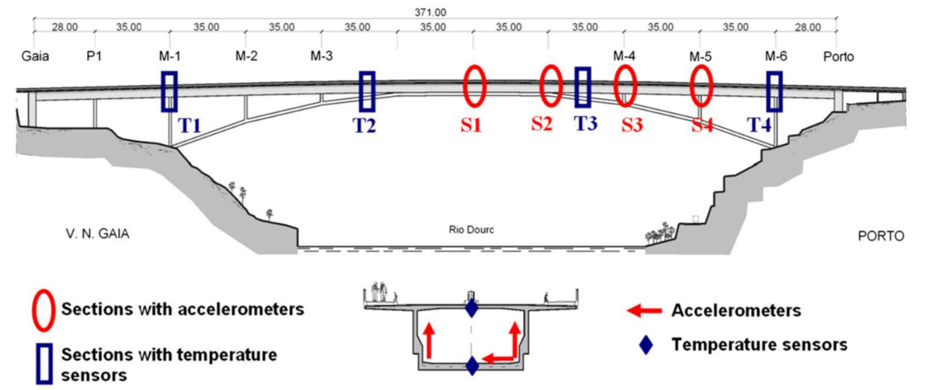


Fig. 2. Position of the accelerometers and temperature sensors.

Source : Magalhães, Filipe, A. Cunha, and Elsa Caetano. "Vibration based structural health monitoring of an arch bridge: from automated OMA to damage detection." *Mechanical Systems and Signal Processing* 28 (2012): 212-228.

OPERATIONAL MODAL ANALYSIS

EXAMPLE ON OMA

OMA allows to continuously obtain an estimate of the resonance frequencies and damping ratio's of the bridge.

The natural vibration is sufficient, no additional excitation is required.

Temperature dependency of the bridge dynamics can be clearly identified.

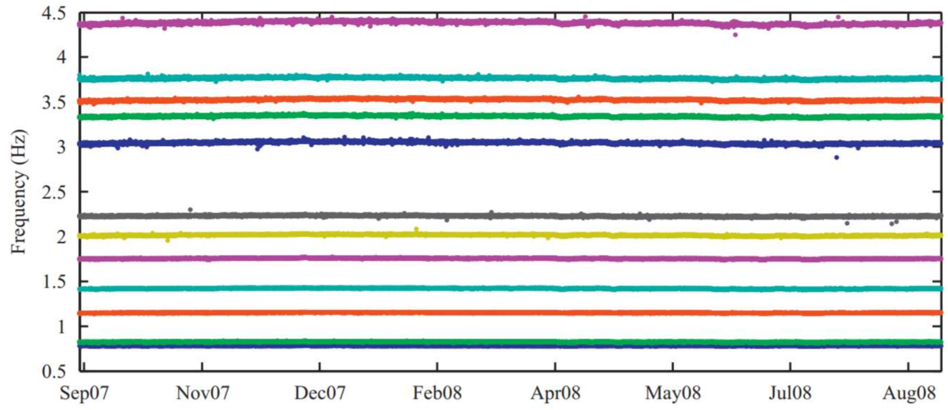


Fig. 3. Time evolution of the bridge first 12 natural frequencies identified with the p-LSCF method, from 13/09/2007 to 12/09/2008.

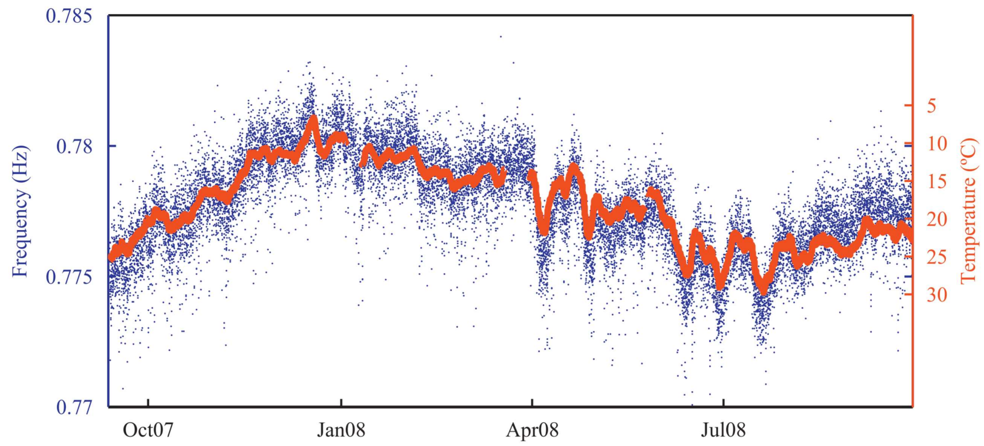
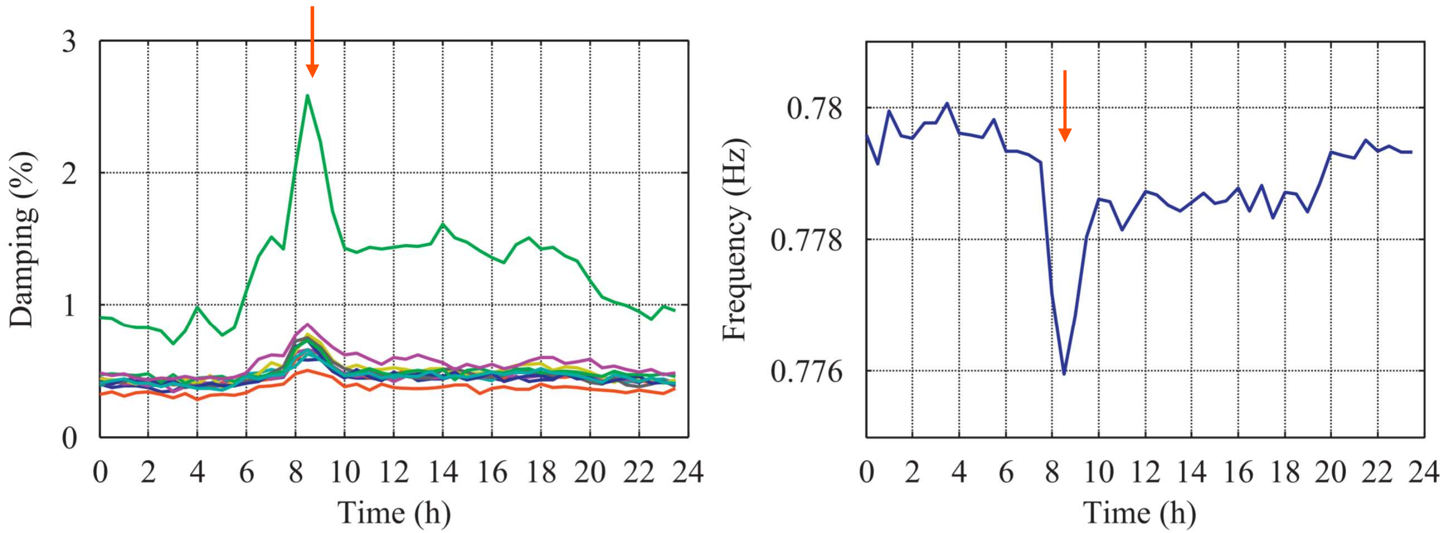


Fig. 4. Time evolution of the first natural frequency (dots) vs. temperature at the top of section *T*3 (line).

OPERATIONAL MODAL ANALYSIS

EXAMPLE ON OMA

But even more interesting :



Daily traffic jam on the bridge:
- **Increased mass** -> Reduced frequency
- **Cars act as 'dampers'** –> Increased damping

Fig. 6. Average day evolution of the modal damping ratios (second mode in green) and of the natural frequency of the first mode during the working days of November 2007. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

OPERATIONAL MODAL ANALYSIS

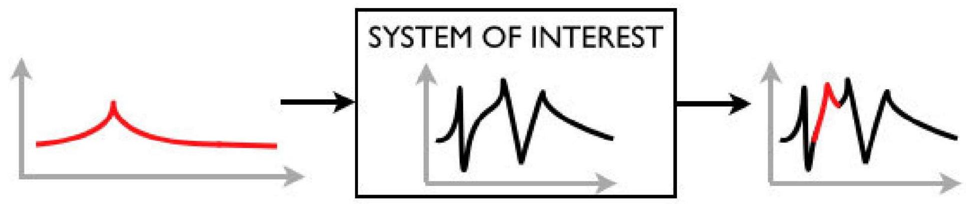
BUT WHAT IF THE INPUT FORCE IS NOT WHITE

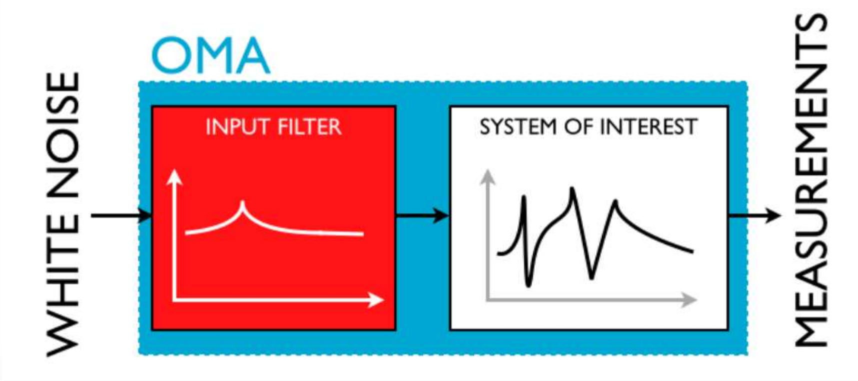
When the input force contains a coloration:

- Periodic load : e.g. due to a rotating component
- Non white load : e.g. People jumping in a grand-stand

$$\widehat{\bar{\bar{G}}_{XX}}(\omega_i) \cancel{=} \bar{\bar{H}}(\omega_i)\bar{\bar{H}}^H(\omega_i)$$

OMA will mis-interpret the input coloration as part of the system.

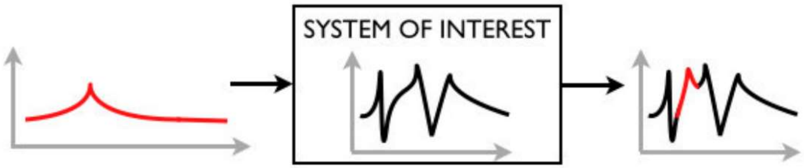




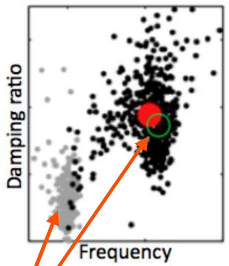
OPERATIONAL MODAL ANALYSIS

BUT WHAT IF THE INPUT FORCE IS NOT WHITE

This *misinterpretation* does not necessarily have to impede the results of OMA.



(a)



(b)

(a) Non-white loads well distanced from the structural modes will, without prior knowledge, be mistaken for structural modes.
(b) Estimated poles of a 500 run simulation for OMA show two distinctive clusters, one is related to the structural pole (○- true value, ●- estimated pole, ●- mean result over 500 runs) the other to the input filter/periodical load (●). When both the input filter as well as the actual pole are estimated, the error upon the estimated poles remains relatively small.

OMA identifies both modes, we should ignore the *input* mode

OPERATIONAL MODAL ANALYSIS

BUT WHAT IF THE INPUT FORCE IS NOT WHITE

This *misinterpretation* does not necessarily have to impede the results of OMA.

But we should be careful when there is a strong interaction between input force and the structural dynamics (e.g. resonating behaviour)

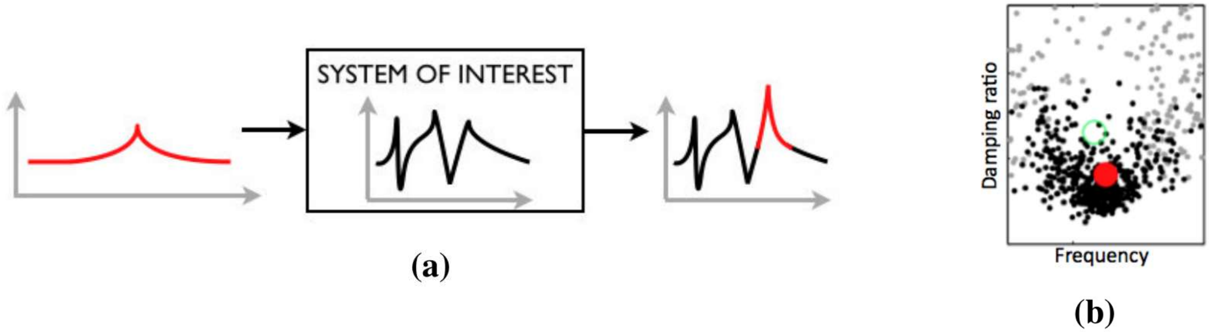


Figure 4.4: (a) Periodical loads close to structural modes are no longer distinguishable from each other.
(b) System poles of 500 run simulation for OMA, during which OMA is no longer able to distinct between the input filter and the system pole itself, causing a severe underestimation of the structural damping.

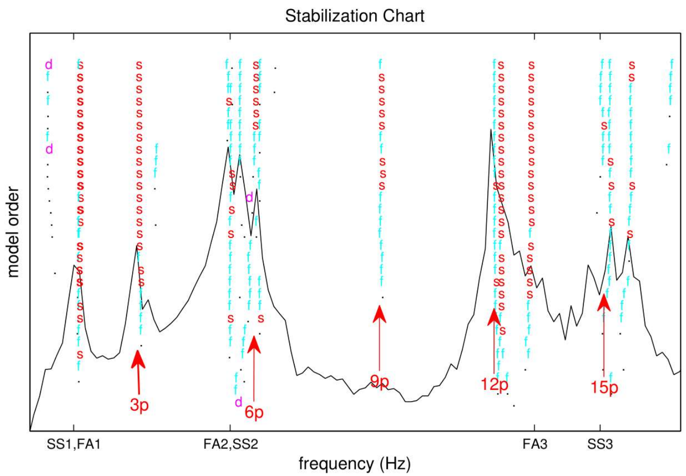
OPERATIONAL MODAL ANALYSIS

EXAMPLE

Wind turbines have a recurring periodical load as the turbine rotates on top.

The rotor generates a set of periodic loads, referred to as harmonics. These are related to the rotational speed of the rotor.

$$f_{1P} = \frac{Rotor\ speed\ (RPM)}{60}$$



Stabilization chart for a dataset during which the turbine was constantly rotating at 16rpm which yields harmonics at 0.8, 1.6, 2.4, 3.2 and 4Hz.

WAIT A SECOND, SO WE HAVE THESE INPUTS WITH A VARYING FREQUENCY?

THAT SOUNDS FAMILIAR! CAN'T WE EXPLOIT THAT?

QUESTIONS?