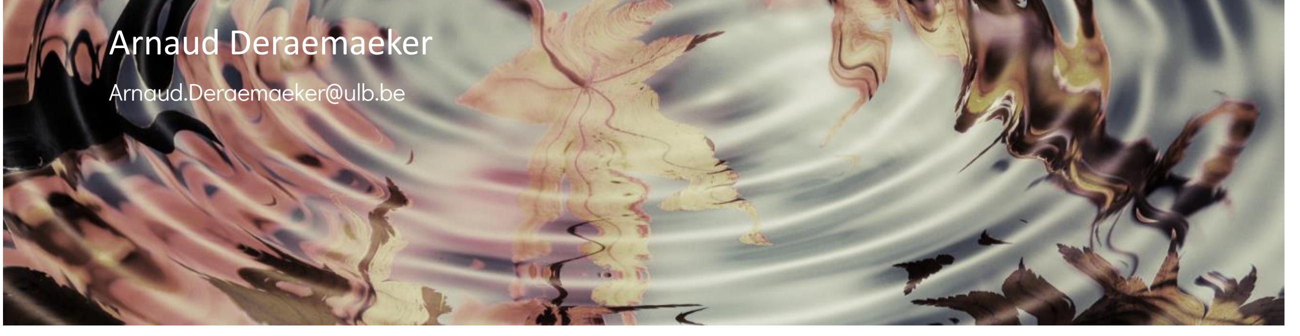
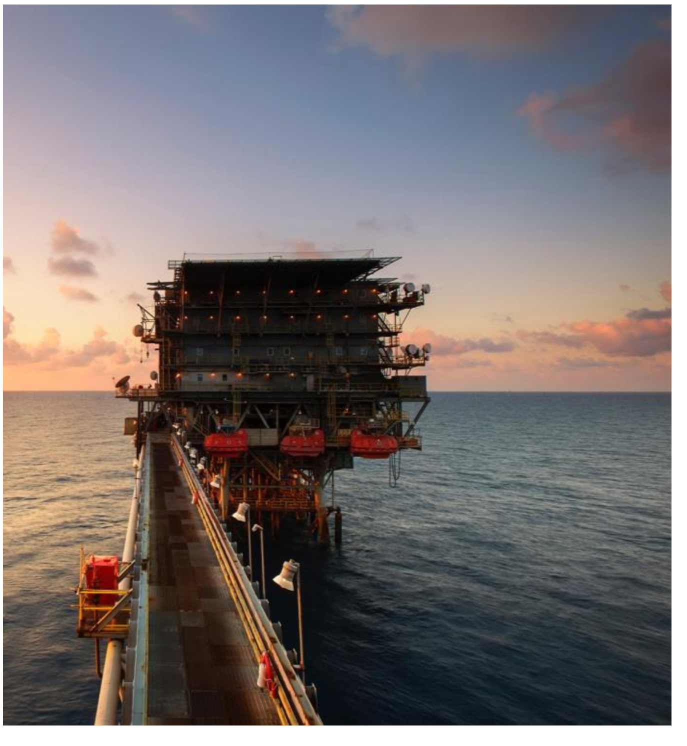
ONE DEGREE OF FREEDOM SYSTEMS



Arnaud Deraemaeker
Arnaud.Deraemaeker@ulb.be

1

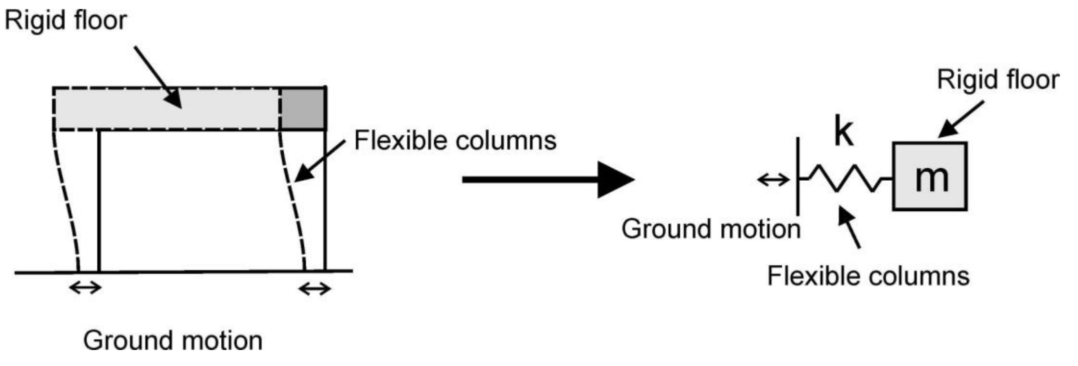
ONE DEGREE OF FREEDOM SYSTEMS IN REAL LIFE



2

Reduction of a system to a one DOF system

Example 1:

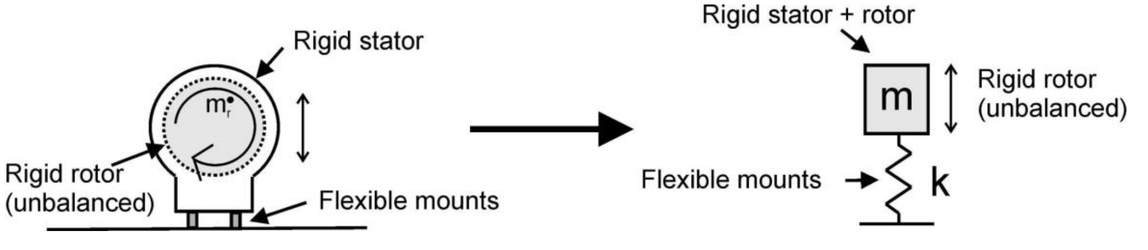


3

3

Reduction of a system to a one DOF system

Example 2:

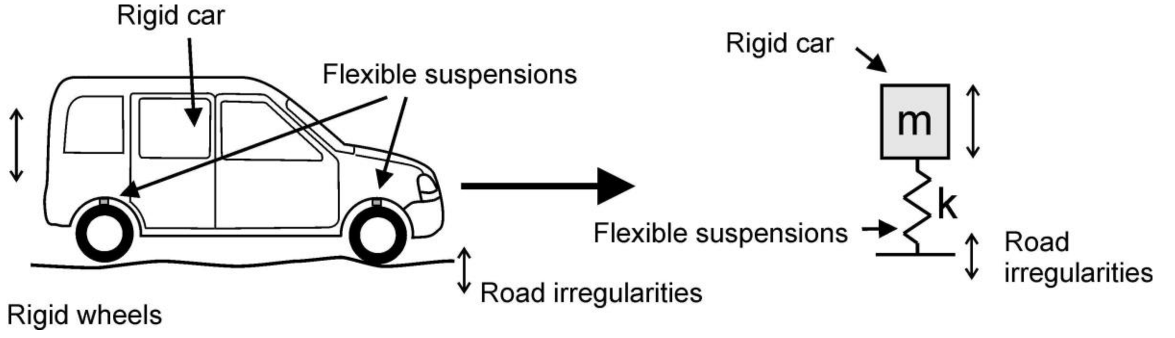


4

4

Reduction of a system to a one DOF system

Example 3:

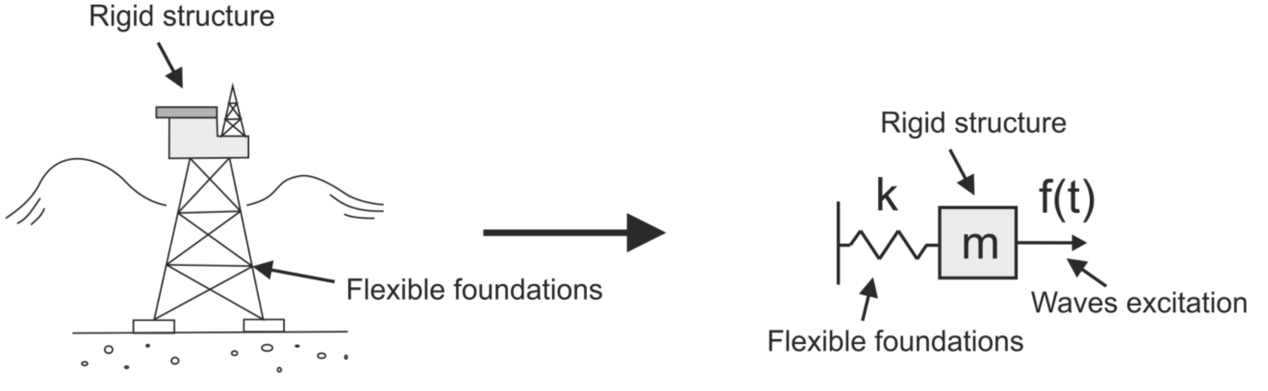


5

5

Reduction of a system to a one DOF system

Example 4:



Offshore platform

6

6

HARMONIC MOTION

7

Simple harmonic motion

https://www.youtube.com/watch?v=gZ_KnZHCn4M

8

8

Harmonic signals

A periodic vibration of which the amplitude can be described by a sinusoidal function:

$$u(t) = a\cos(\omega t + \phi)$$

$$u(t) = a\sin(\omega t + \phi)$$

is called an harmonic vibration
with:

- amplitude a
- angular frequency $\omega = 2\pi f$
- frequency f
- period T = 1/f or f = 1/T
- phase angle ϕ at t=0
- total phase angle $\omega t + \phi$

9

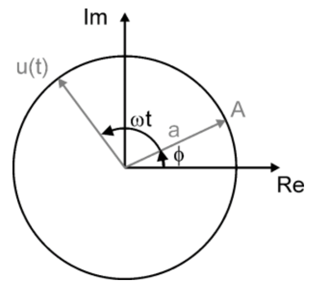
9

Harmonic signals

Representation in the complex plane:

$$\begin{aligned} u(t) &= ae^{i(\omega t+\phi)} \\ &= a\cos(\omega t + \phi) + ia\sin(\omega t + \phi) \end{aligned}$$

$$\begin{aligned} u(t) &= ae^{i\phi}e^{i\omega t} = Ae^{i\omega t} \\ A &= \underbrace{a\cos\phi + ia\sin\phi}_{\text{Independent of time}} \end{aligned}$$



Projection of the rotating vector on the real axis is a cosine

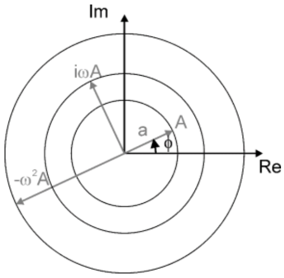
Projection of the rotating vector on the imaginary axis is a sine

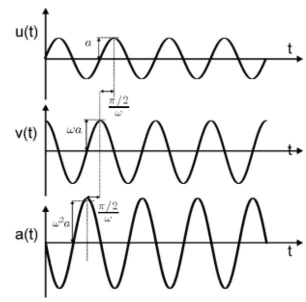
10

10

Harmonic signals

$$
\begin{aligned}
\text{Displacement} \quad u(t) &= Ae^{i\omega t} \\
\text{Velocity} \quad v(t) &= \frac{d\,u(t)}{dt} = i\omega Ae^{i\omega t} = i\omega u(t) \\
\text{Acceleration} \quad a(t) &= \frac{d\,v(t)}{dt} = -\omega^2 Ae^{i\omega t} = -\omega^2 u(t)
\end{aligned}
$$



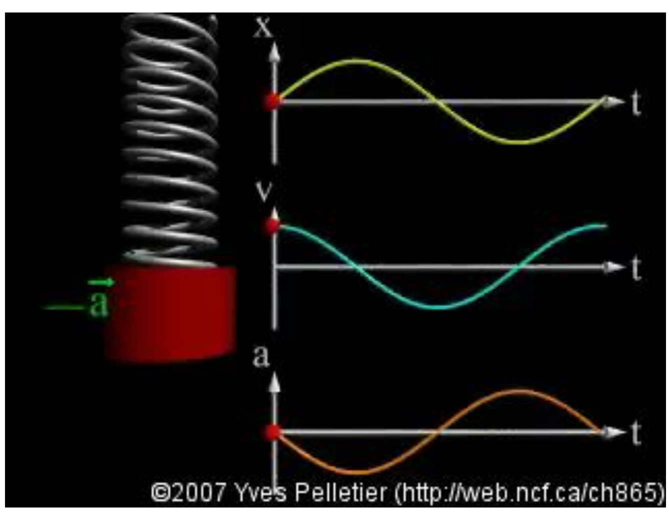


the phase angle of u(t) is 90° behind v(t)

the phase angle of v(t) is 90° behind a(t)

11

Harmonic signals



https://youtu.be/eeYRkW8V7Vg

12

UNDAMPED VIBRATIONS OF A ONE DOF SYSTEM

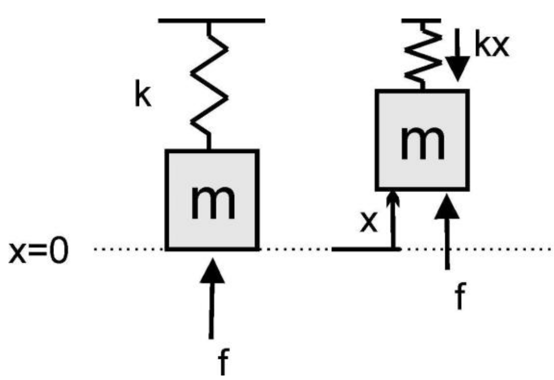
13

Equation of motion

Newton's law:

- Spring force: $-kx$
- External force f acting on the mass.

$$m\ddot{x} = \sum F_x \longrightarrow \boxed{m\ddot{x} + kx = f}$$

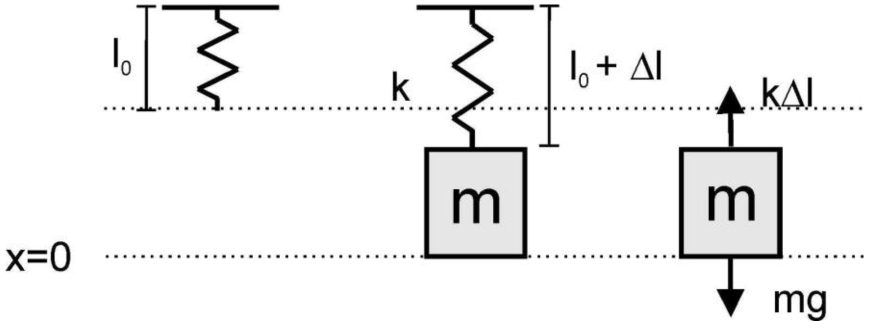


14

14

Equation of motion

What about the effect of gravity ?



The displacement x is defined with respect to the equilibrium position of the mass subjected to gravity. The effect of gravity should therefore not be taken into account in the equation of motion of the system.

15

Free vibrations

$$\boxed{m\ddot{x} + kx = 0} \qquad x = A\,e^{rt}$$

Characteristic equation:

$$mr^2 + k = 0 \qquad r = \pm i\sqrt{k/m}$$

$$\boxed{x = A\cos\omega_n t + B\sin\omega_n t} \qquad \omega_n = \sqrt{k/m}$$

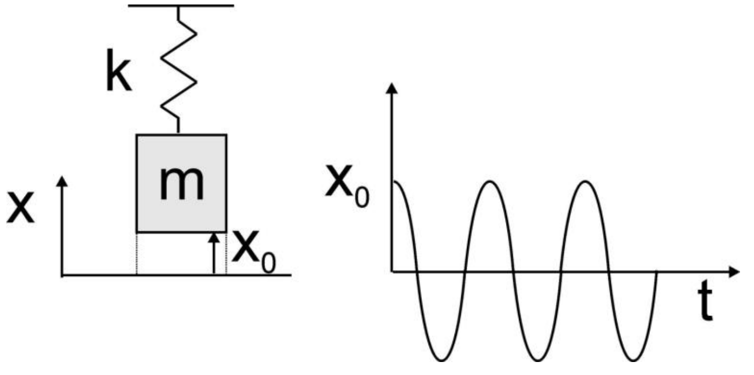
•In the absence of external excitation force, the motion is oscillatory. The natural angular frequency ω_n is defined by the values of k and m
•The motion is initialized by imposing initial conditions on the displacement and the velocity

16

Free vibrations

displacement x_0 velocity $\dot{x}_0$

$$\boxed{x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t}$$



17

17

Free vibrations

$$\boxed{x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t}$$

Alternative representation:

$$x(t) = a \cos (\omega_n t + \phi)$$

$$x_0 = a \cos \phi \qquad \tan \phi = \frac{\dot{x}_0}{\omega_n x_0}$$

$$\dot{x}_0 / \omega_n = a \sin \phi$$

The motion can be described by a cosine function with a phase. The phase is a function of the initial conditions.

18

18

Harmonic excitation

$$f(t) = F\,e^{i\omega t} \qquad x(t) = X\,e^{i\omega t}$$

$$m\ddot{x} + kx = f$$

$$\left(k - \omega^2 m\right) X e^{i\omega t} = F e^{i\omega t}$$

$$\boxed{X = \frac{F}{k - \omega^2 m}}$$

19

19

Harmonic excitation

$$X = \frac{F}{k - \omega^2 m}$$

$$X_0 = F/k \quad (\omega = 0)$$

$$\omega_n = \sqrt{k/m}$$

$$\boxed{\frac{X}{X_0} = \frac{1}{1 - \omega^2/\omega_n^2}}$$

- Positive if $\omega < \omega_n$
- Infinite if $\omega = \omega_n$
- Negative if $\omega > \omega_n$

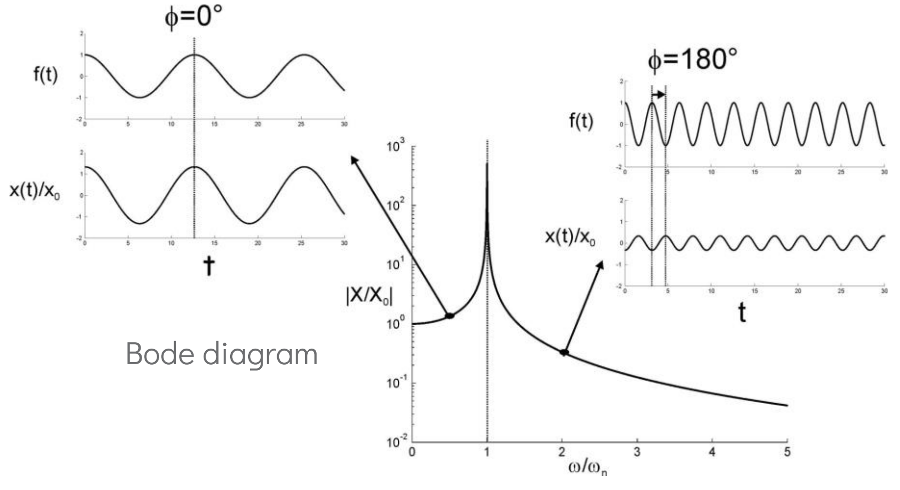
20

20

Harmonic excitation

$$\boxed{\frac{X}{X_0} = \frac{1}{1 - \omega^2/\omega_n^2}}$$

Dynamic amplification factor



Bode diagram

21

21

RESONANCE

22

Breaking a glass of wine with sound

https://www.youtube.com/watch?v=10lWpHyN0Ok

23

23

Breaking a glass of wine with sound

https://www.youtube.com/watch?v=JiM6AtNLXX4

24

Is stiffer stronger ?

https://youtu.be/n9ULMIjvSIg

25

25

Is stiffer stronger ?

https://youtu.be/LV_UuzEznHs

26

26

Buildings resonance

https://youtu.be/pMr1MzSv044

27

27

DAMPED VIBRATIONS OF A ONE DOF SYSTEM

28

Effect of damping on a building

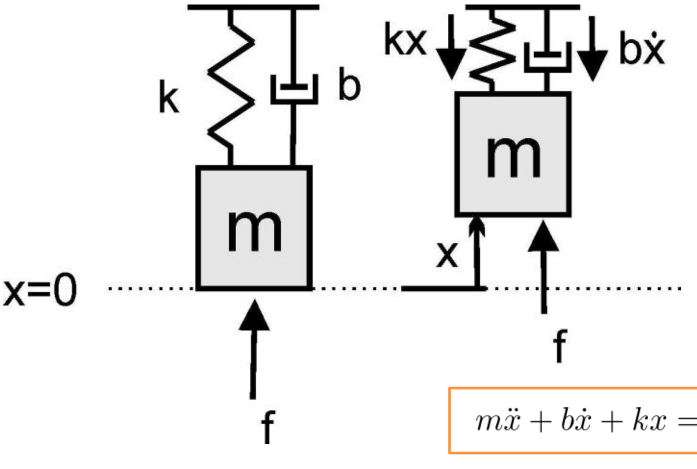
https://youtu.be/HWpkaIB1fD0

29

29

Equation of motion

Damping force : $F_b = -b\dot{x}$ viscous damping



$$\boxed{m\ddot{x} + b\dot{x} + kx = f}$$

30

30

Free vibrations

$$m\ddot{x} + b\dot{x} + kx = f$$

$$\downarrow$$

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = f/m$$

$$\xi = \frac{b}{2\sqrt{km}}$$ damping coefficient

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0$$ General solution $x = A\,e^{rt}$

Characteristic equation:

$$r^2 + 2\xi\omega_n r + \omega_n^2 = 0$$

$$\xi < 1$$

$$w_d = \omega_n\sqrt{1-\xi^2}$$

$$\downarrow$$

$$r = -\xi\omega_n \pm i\omega_n\sqrt{1-\xi^2} = -\xi\omega_n \pm i\omega_d$$

31

31

Free vibrations

$$x(t) = e^{-\xi\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

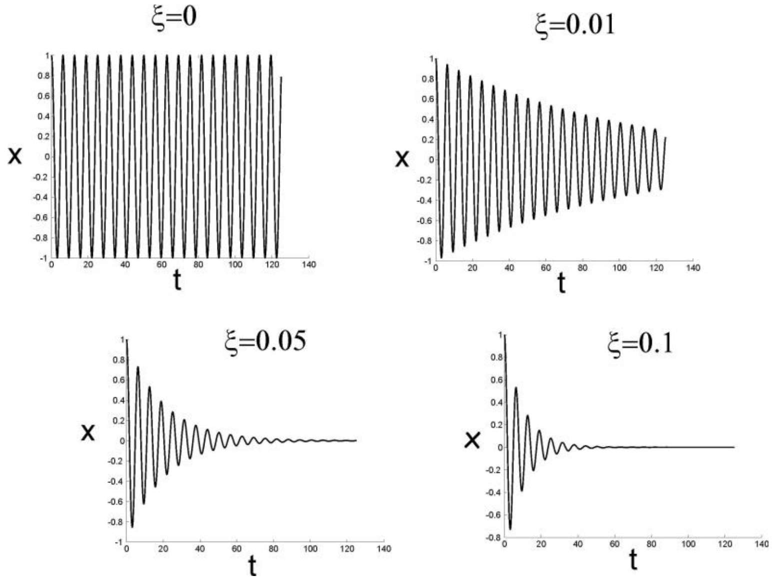
Initial conditions: displacement x_0 velocity $\dot{x_0}$

$$\boxed{x(t) = e^{-\xi\omega_n t}\left(x_0\cos\omega_d t + \frac{\dot{x_0} + \omega_n\xi x_0}{\omega_d}\sin\omega_d t\right)}$$

32

32

Free vibrations

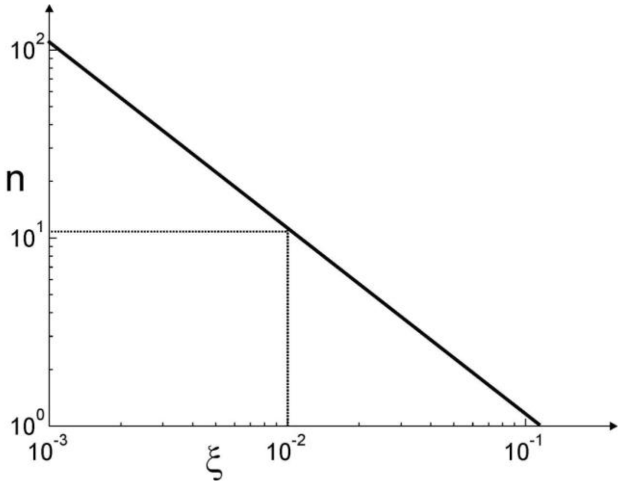


Free response to unit initial displacement

33

33

Free vibrations



Number of oscillations after which the vibration amplitude is reduced by one half

34

34

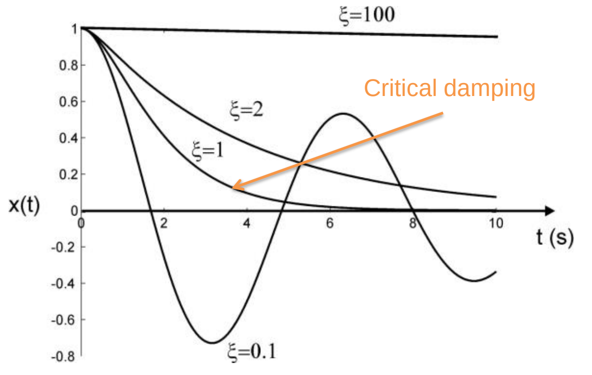
Free vibrations

$$\xi > 1 \qquad x(t) = e^{-\xi\omega_n t}\left(x_0 \cosh \mu t + \frac{\dot{x_0} + \omega_n \xi x_0}{\mu} \sinh \mu t\right)$$

$$\mu = \omega_n \sqrt{\xi^2 - 1}$$

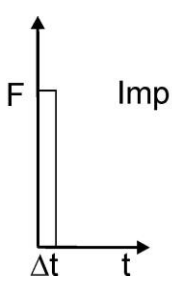
$$\xi = 1 \qquad x(t) = e^{-\omega_n t}\left((\dot{x_0} + \omega_n x_0)\, t + x_0\right)$$

Critical damping



35

Impulse response



Impulse=FΔt

$$m\ddot{x} + b\dot{x} + kx = f \qquad x_0 = 0,\ \dot{x_0} = 0$$

$$m\dot{x_0}|_{\Delta t} = F\Delta t - \cancel{\int_0^{\Delta t} kx\,dt} - \cancel{\int_0^{\Delta t} b\dot{x}\,dt}$$

$$\boxed{\dot{x_0}|_{\Delta t} = \frac{F\Delta t}{m}}$$

Equivalent to initial velocity at Δt

36

Impulse response

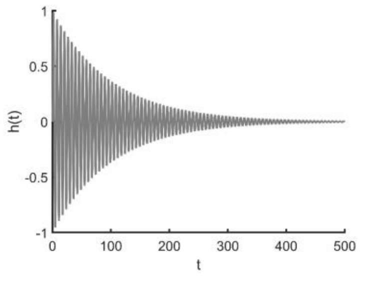
For an initial velocity, the response of the system is:

$$x(t) = \frac{e^{-\xi\omega_n t}\dot{x_0}}{\omega_d}\sin(\omega_d t)$$

with $\dot{x_0} = \frac{F\Delta t}{m}$

For a unit impulse $F\Delta t = 1$, we define the impulse response *h(t)*

$$\boxed{h(t) = \frac{e^{-\xi\omega_n t}}{m\omega_d}\sin(\omega_d t)}$$



$\omega_n = 1, \xi = 0.01$

37

37

Harmonic excitation

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = f/m$$

$$x(t) = X\,e^{i\omega t}$$

$$f(t) = F\,e^{i\omega t}$$

$$(\omega_n^2 + 2i\xi\omega\omega_n - \omega^2)X = F/m$$

$$X = \frac{F}{m}\left(\frac{1}{\omega_n^2 + 2i\xi\omega\omega_n - \omega^2}\right) = \frac{F}{k}\left(\frac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2i\xi\frac{\omega}{\omega_n}}\right)$$

$$= X_0\left(\frac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2i\xi\frac{\omega}{\omega_n}}\right)$$

38

38

Harmonic excitation

$$X_r = X_0 \frac{1 - \frac{\omega^2}{\omega_n^2}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}$$

$$X_i = X_0 \frac{-2\xi \frac{\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}$$

$$|X/X_0| = \sqrt{\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

$$\tan\phi = \frac{-2\xi \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

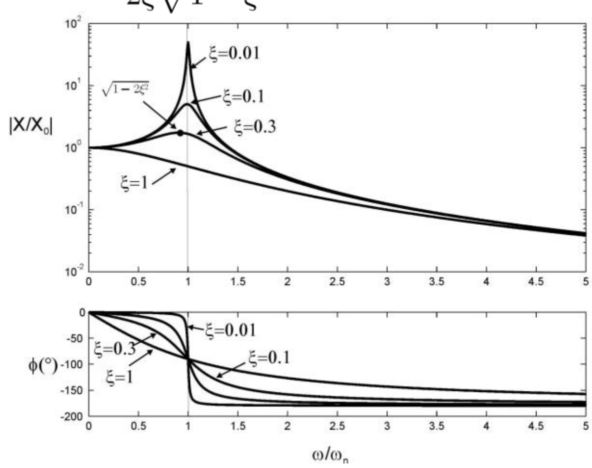
39

39

Harmonic excitation

Maximum $\omega/\omega_n = \sqrt{1 - 2\xi^2}$

Max ampl. $|X/X_0| = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$



For ξ small

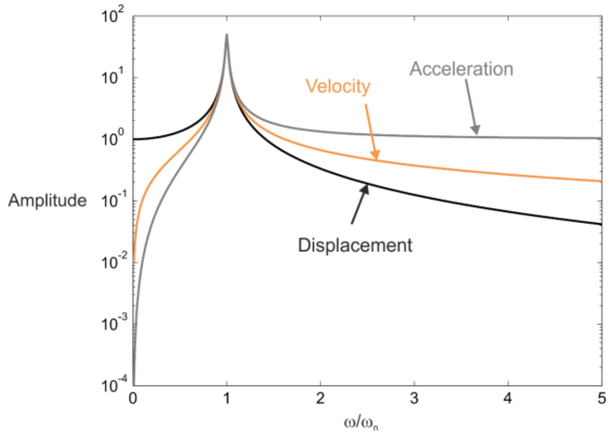
$$\begin{aligned} \omega/\omega_n &= 1 \\ |X/X_0| &= \frac{1}{2\xi} \end{aligned}$$

40

40

Harmonic excitation

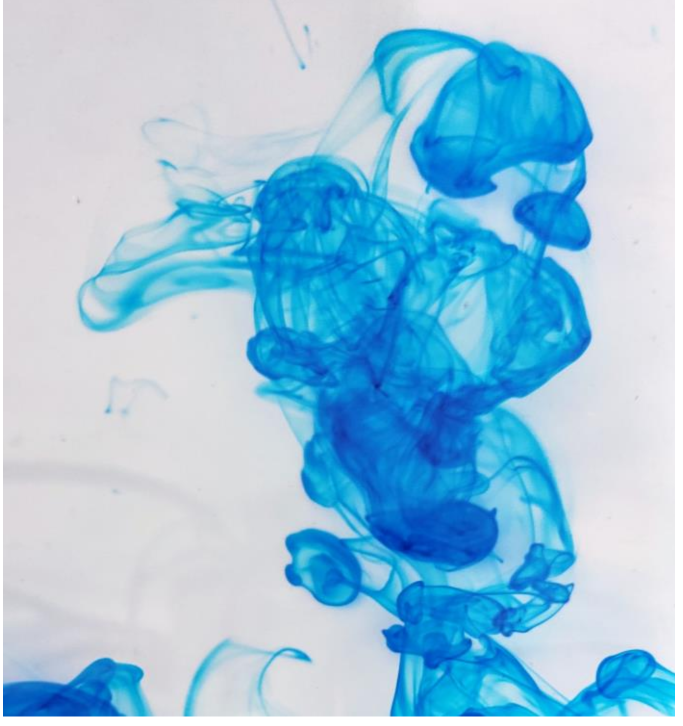
$$\begin{aligned}
\text{Displacement} \quad u(t) &= Ae^{i\omega t} \\
\text{Velocity} \quad v(t) &= \frac{d\,u(t)}{dt} = i\omega Ae^{i\omega t} = i\omega u(t) \\
\text{Acceleration} \quad a(t) &= \frac{d\,v(t)}{dt} = -\omega^2 Ae^{i\omega t} = -\omega^2 u(t)
\end{aligned}$$



41

41

DYNAMIC RESPONSE FOR ARBITRARY FORCE INPUT



42

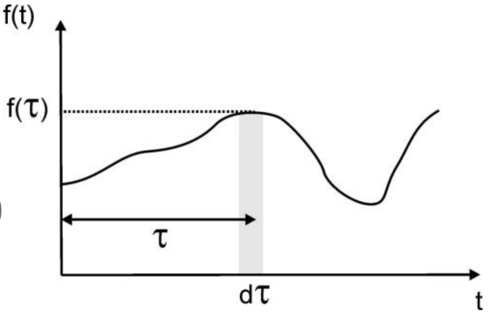
Duhamel's integral

f(t) is decomposed into a series of short impulses at time t

The contribution of one impulse $f(\tau)d\tau$ to the response of the system is given by :

$$f(\tau)d\tau h(t-\tau)$$

(h(t) is the impulse response)



The total contribution is therefore:

$$x(t) = \int_0^t f(\tau)h(t-\tau)d\tau$$

We have $h(t) = 0$ and $f(t) = 0$ for $t < 0$ so that we can write

$$x(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = f(t) * h(t)$$

43

43

Convolution integral

The convolution integral of two time functions *x(t)* and *h(t)* yields a new time function *y(t)* defined as:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = x(t) * h(t)$$

- Take the two functions *x(t)* and *h(t)* and replace t by the dummy variable τ
- Mirror the function *h(τ)* against the ordinate, this yields *h(-τ)*
- Shift the function *h(-τ)* with a quantity *t*
- Determine for each value of t the product of *x(τ)* with *h(t-τ)*
- Compute the integral of the product *y(t)*
- let τ vary from $-\infty$ (or a value small enough to make the product zero) to ∞ (or a value of t that is big enough)

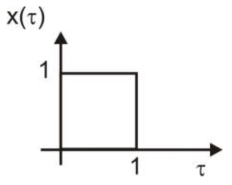
44

44

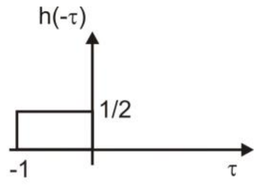
Convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

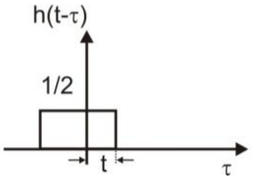
1st function



Mirrored 2nd function

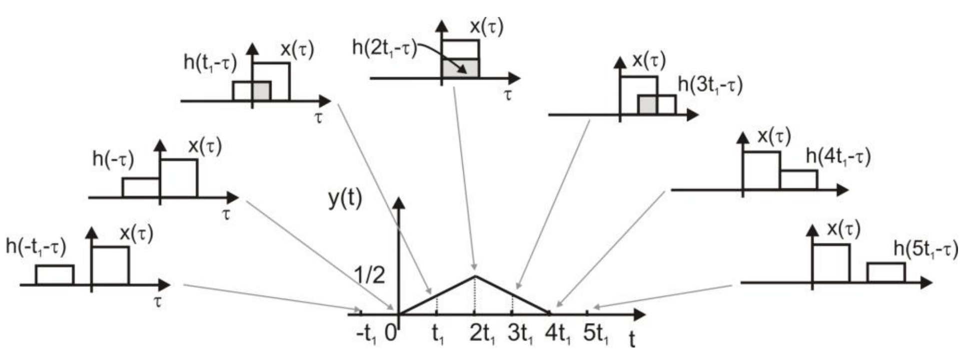


Shift h(t):



Convolution integral

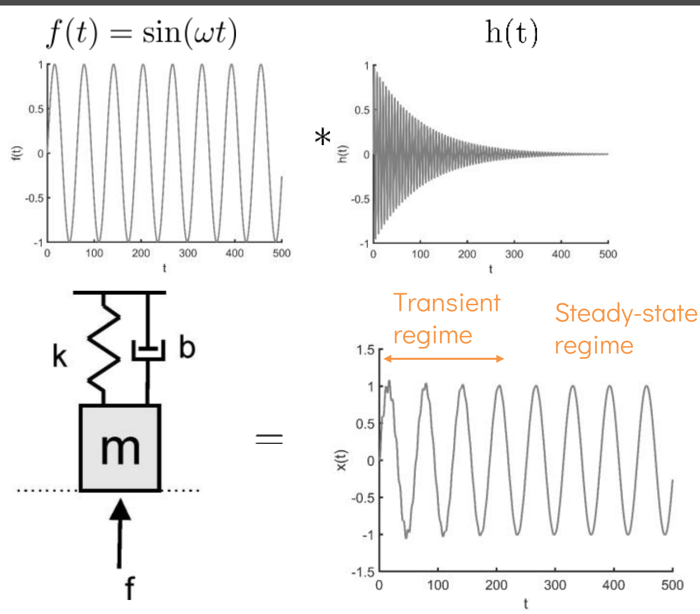
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



Property:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

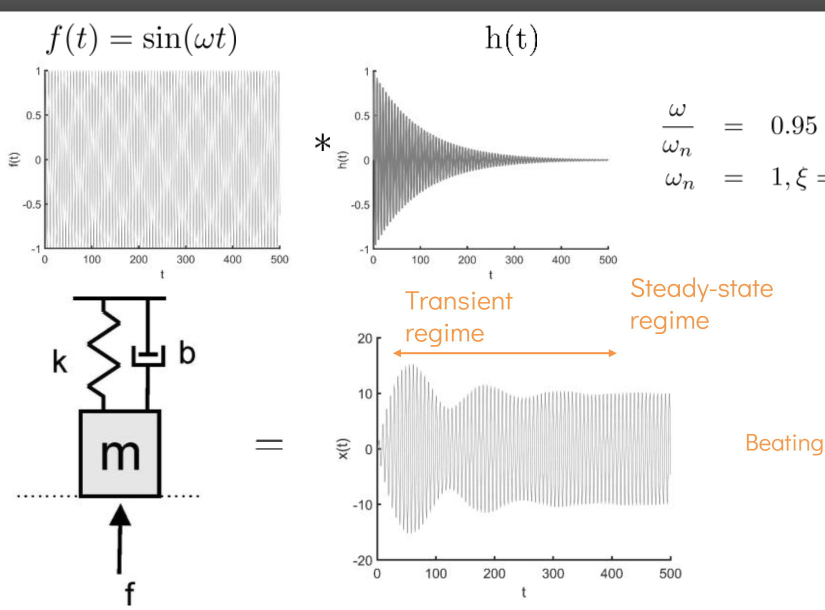
Harmonic excitation below resonance



$$\begin{aligned}\frac{\omega}{\omega_n} &= 0.1\\ \omega_n &= 1, \xi = 0.01\end{aligned}$$

47

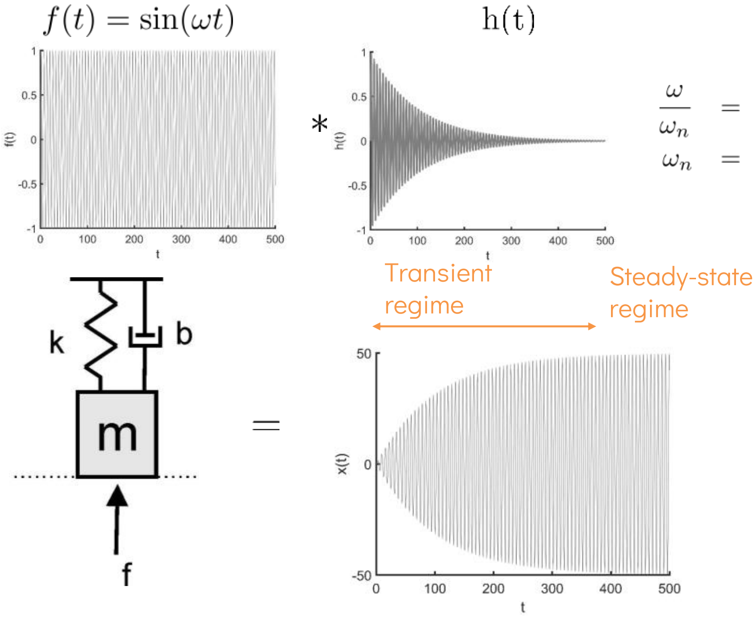
Harmonic excitation near resonance



$$\begin{aligned}\frac{\omega}{\omega_n} &= 0.95\\ \omega_n &= 1, \xi = 0.01\end{aligned}$$

48

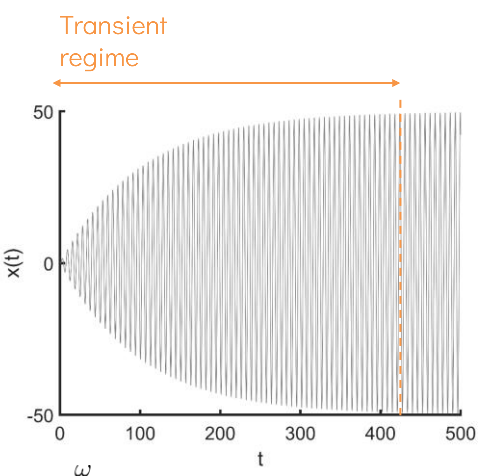
Harmonic excitation at resonance



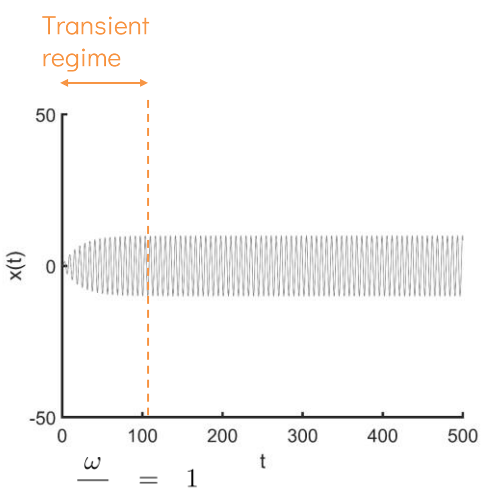
$$\frac{\omega}{\omega_n} = 1$$
$$\omega_n = 1, \xi = 0.01$$

49

Harmonic excitation at resonance



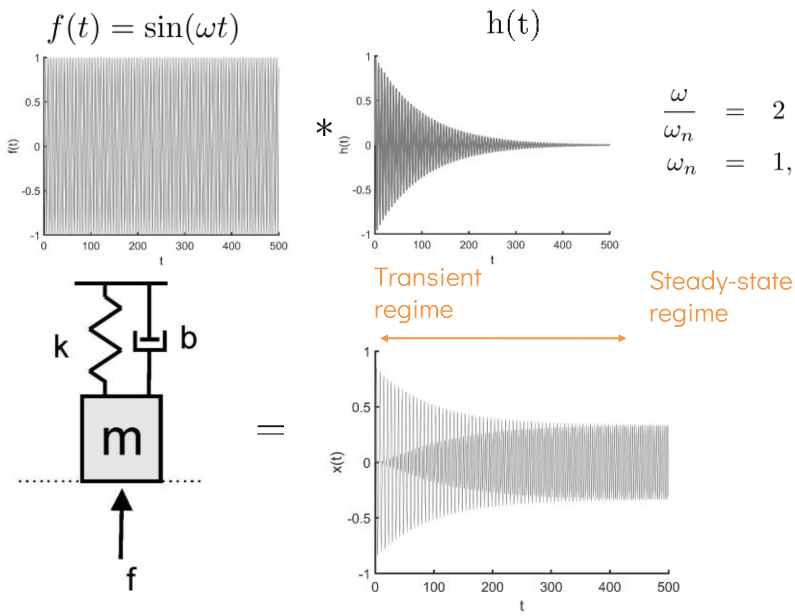
$$\frac{\omega}{\omega_n} = 1$$
$$\omega_n = 1, \xi = 0.01$$



$$\frac{\omega}{\omega_n} = 1$$
$$\omega_n = 1, \xi = 0.05$$

50

Harmonic excitation above resonance

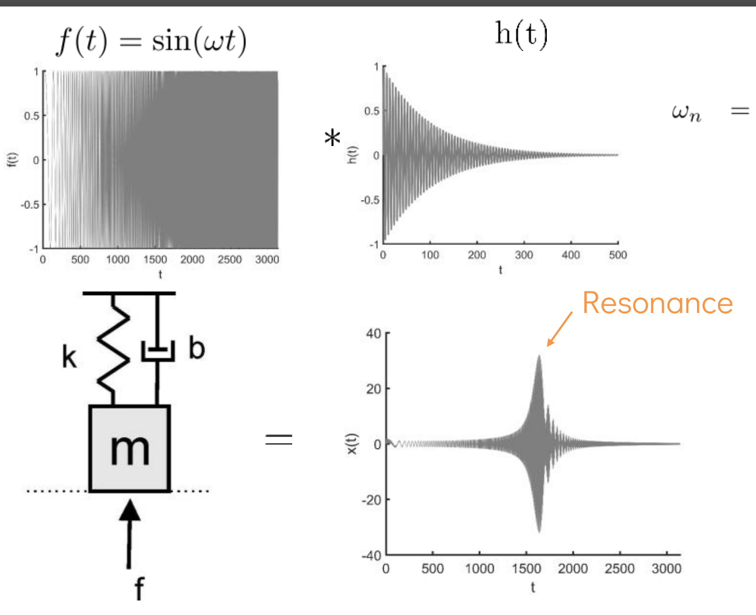


$$\frac{\omega}{\omega_n} = 2$$
$$\omega_n = 1, \xi = 0.01$$

51

51

Sine sweep excitation



$$\omega_n = 1, \xi = 0.01$$

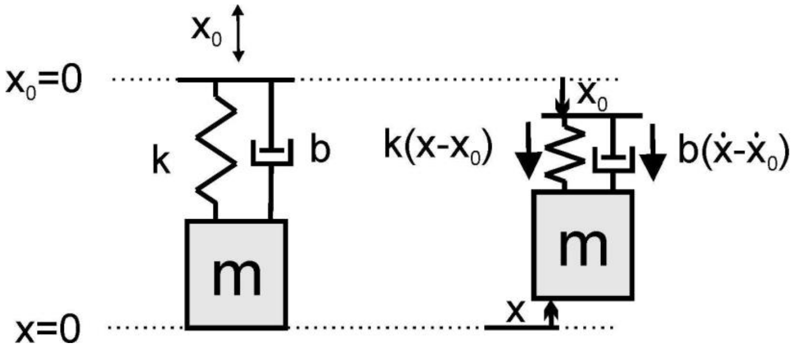
52

52

DYNAMIC RESPONSE TO BASE EXCITATION

53

Equation of motion



$$m\ddot{x} = -k(x - x_0) - b(\dot{x} - \dot{x}_0)$$

$$x_r = x - x_0$$

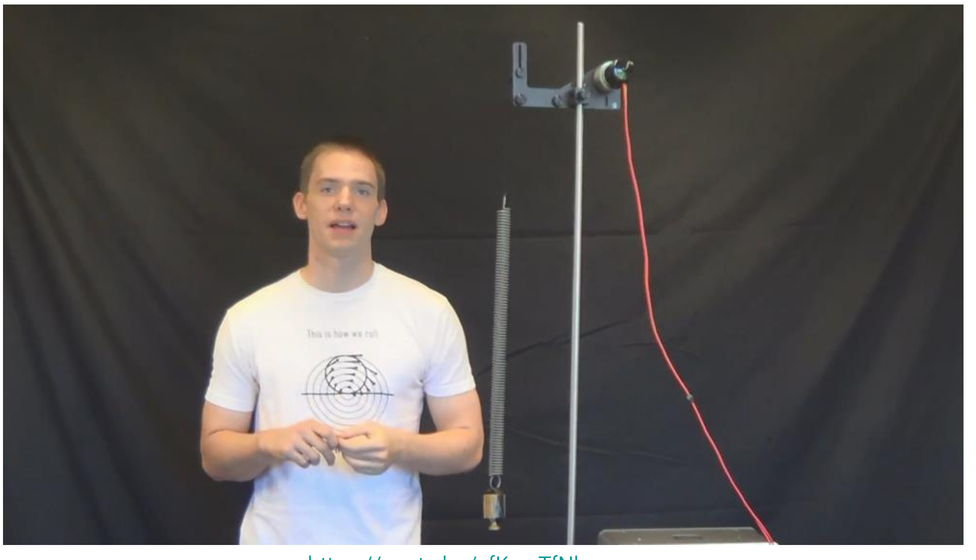
$$m\ddot{x}_r + b\dot{x}_r + kx_r = -m\ddot{x}_0$$

Excitation

54

54

Harmonic response

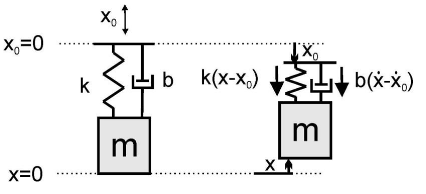


https://youtu.be/cfKwnTfNhog

55

55

Response to ground acceleration in the time domain



$$m\ddot{x} = -k(x - x_0) - b(\dot{x} - \dot{x}_0)$$
$$x_r = x - x_0$$
$$m\ddot{x}_r + b\dot{x}_r + kx_r = -m\ddot{x}_0$$

Earthquake base acceleration

Duhamel's integral

$$x(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = f(t) * h(t)$$

Impulse response

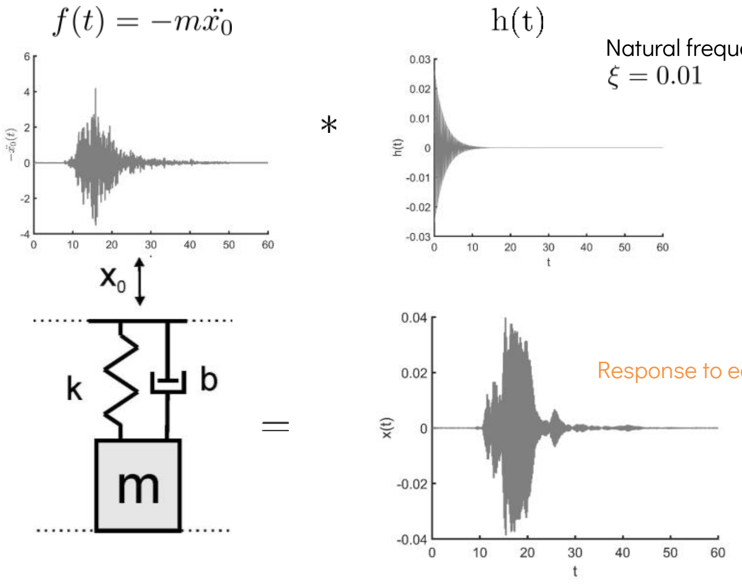
$$h(t) = \frac{e^{-\xi\omega_n t}}{m\omega_d}\sin(\omega_d t)$$

$$\longrightarrow \quad x_r(t) = \int_{-\infty}^{\infty} -\ddot{x}_0(\tau)\frac{e^{-\xi\omega_n(t-\tau)}}{\omega_d}\sin(\omega_d(t-\tau))d\tau$$

56

56

Santa Cruz earthquake (1990)



Natural frequency = 6 Hz

$\xi = 0.01$

Response to earthquake excitation

57

57