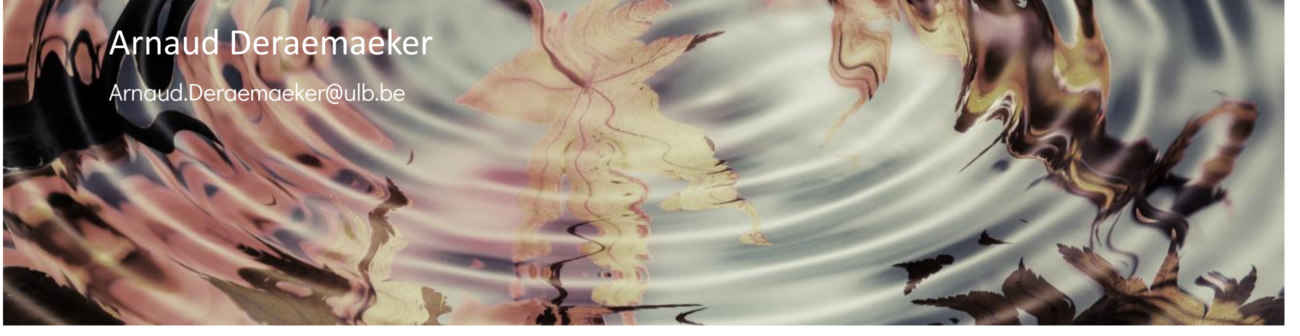
VIBRATION SOURCES



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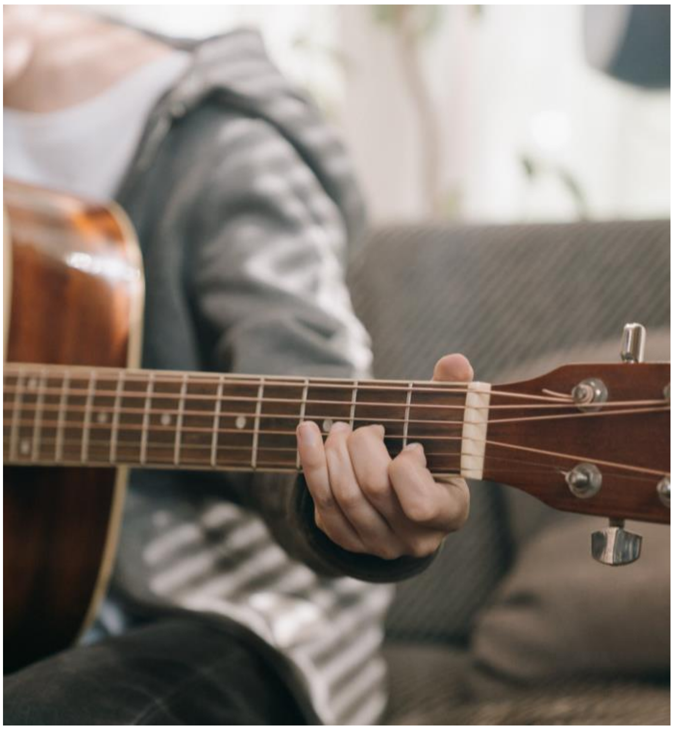
FREE AND FORCED VIBRATIONS



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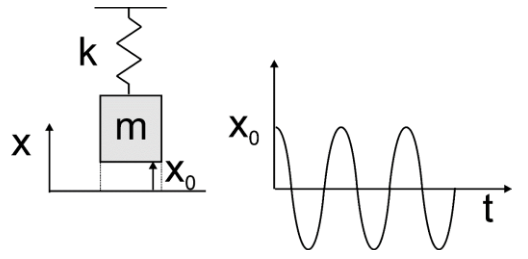
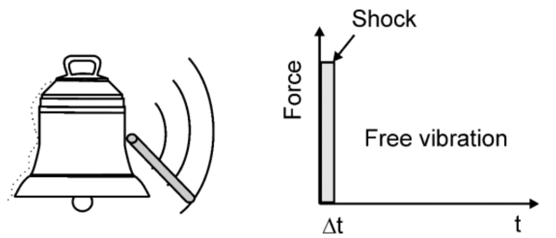
Free vibrations

Short initial excitation

Initial displacement



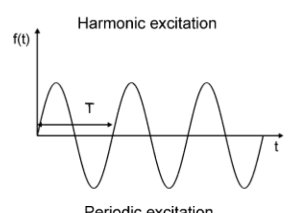
Shock



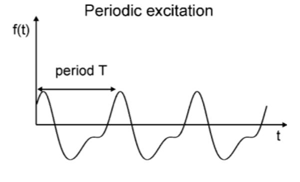
Forced vibrations

Continuous excitation

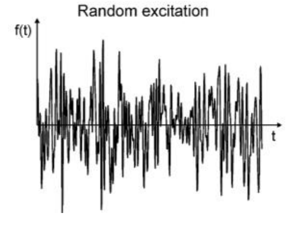
Harmonic force signal



Periodic force signal

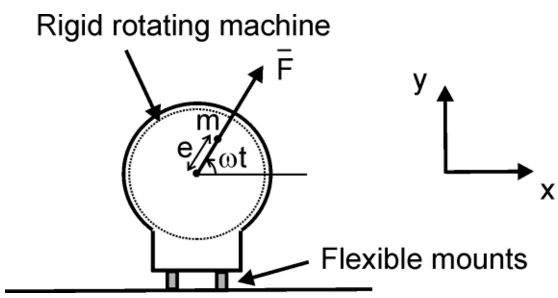


Random force signal



Harmonic excitation

The signal is in the form of a sine or/and cosine function



$$F_x = me\omega^2 \cos(\omega t)$$
$$F_y = me\omega^2 \sin(\omega t)$$

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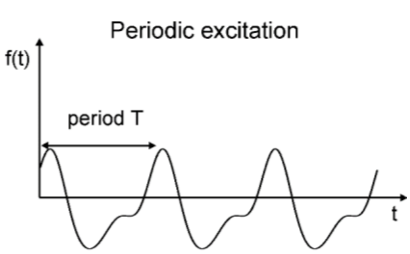
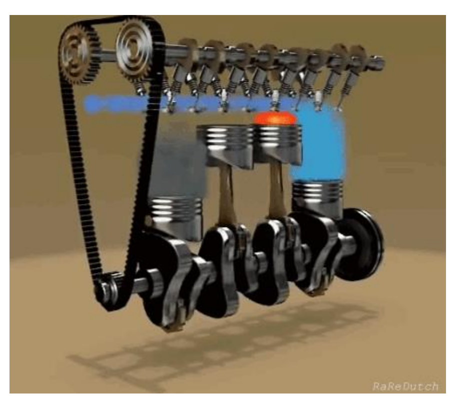
Periodic excitation

The signal repeats itself





Four stroke engine (https://gifer.com)

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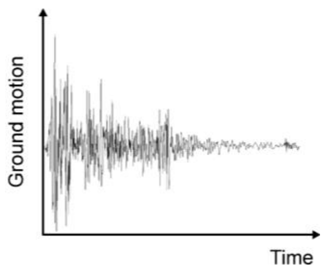
Random excitation

There is no repetition in the signal

- Turbulent wind
- Waves in storms
- Traffic
- Earthquakes



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FREQUENCY CONTENT OF SOURCES

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Example : pedestrian induced vibrations of footbridges

For one pedestrian walking/running at a regular pace, the excitation is periodic

Representative types of activity			Range of applicability		
Designation	Definition	Design activity rate [Hz]	Actual activities	Activity rate [Hz]	Structure type
"walking"	walking, continuous ground contact	1.6 – 2.4	slow walking (ambling) normal walking fast, brisk walking	~ 1.7 ~ 2.0 ~ 2.3	- pedestrian structures (pedestrian bridges, stairs, piers, etc.) - office buildings, etc.
"running"	running, discontinuous ground contact	2.0 – 3.5	slow running (jog) normal running fast running (sprint)	~ 2.1 ~ 2.5 > 3.0	- pedestrian bridges on jogging tracks, etc.

From 'Vibration problems in structures', H. Bachman, 1995

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Example : pedestrian induced vibrations of footbridges

Walking

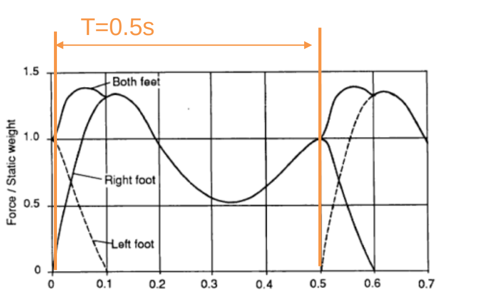

Figure G.1: Forcing function resulting from footfall overlap during walking with a pacing rate of 2 Hz (from [G.4])

Jumping

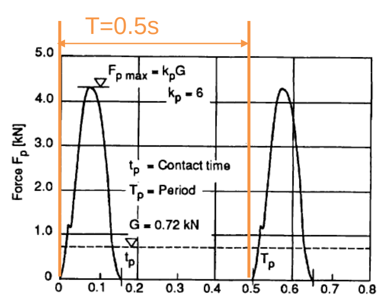

Figure G.2: Forcing function from jumping on the spot with both feet simultaneously with a jumping rate of 2 Hz (from [G.4])

From 'Vibration problems in structures', H. Bachman, 1995

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Example : pedestrian induced vibrations of footbridges

The periodic force acting on the bridge can be expressed as :

$$F_p(t) = G + \sum_{i=1}^{n} G\alpha_i sin(2\pi i f_p t - \phi_i)$$

- G = weight of person
- α_i = Fourier coefficient of the i^{th} harmonic
- $G\ \alpha_i$ = force amplitude of the i^{th} harmonic
- f_p = activity rate (Hz)
- ϕ_i = phase lag of the i^{th} harmonic relative to the first harmonic
- i = number of the i^{th} harmonic
- n = total number of contribution harmonics

From 'Vibration problems in structures', H. Bachman, 1995

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Example : pedestrian induced vibrations of footbridges

Representative types of activity	Activity rate [Hz]		Fourier coefficient and phase lag						Design density [persons/m²]
			α_1	ϕ_1	α_2	ϕ_2	α_3	ϕ_3	
"walking"	vertical	2.0 2.4	0.4 0.5		0.1	$\pi/2$	0.1	$\pi/2$	~ 1
	forward	2.0	0.5($\alpha_{1/2}$=0.1)		0.2				
	lateral	2.0	$\alpha_{1/2}$ = 0.1		$\alpha_{3/2}$ = 0.1				
"running"	2.0 ..... 3.0		1.6		0.7		0.2		-

From 'Vibration problems in structures', H. Bachman, 1995

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THE DISCRETE FOURIER TRANSFORM

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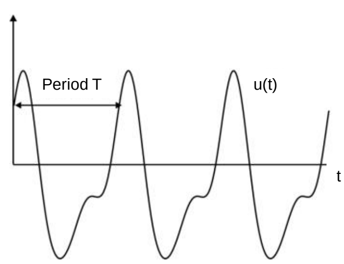
Discrete Fourier series

Let u(t) be a periodic function of period T



u(t) can be decomposed into a discrete Fourier series of the form

$$\boxed{u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right]} \qquad \omega_0 = \frac{2\pi}{T}$$

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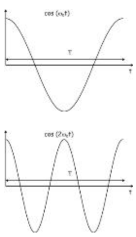
Discrete Fourier series

$\cos(n\omega_0 t)$



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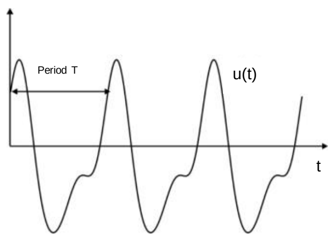
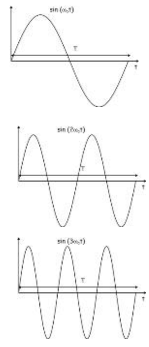


$\sin(n\omega_0 t)$



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Discrete Fourier series

$$\boxed{u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]}$$

$$a_0 = \frac{2}{T}\int_0^T u(t)dt \qquad a_n = \frac{2}{T}\int_0^T u(t)\cos(n\omega_0 t)dt \qquad b_n = \frac{2}{T}\int_0^T u(t)\sin(n\omega_0 t)dt$$

Amplitudes and phases

$$\boxed{u(t) = d_0 + \sum_{n=1}^{\infty} d_n \cos(n\omega_0 t - \phi_n)}$$

$$F_p(t) = G + \sum_{i=1}^{n} G\alpha_i sin(2\pi i f_p t - \phi_i)$$

[Bachman]

$$d_0 = \frac{a_0}{2} \qquad d_n = \sqrt{a_n^2 + b_n^2} \qquad \phi_n = tg^{-1}\left(\frac{b_n}{a_n}\right)$$

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Complex formulation

$$\boxed{u(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{in\omega_0 t}} \qquad c_0 = \frac{a_0}{2} = \frac{1}{T}\int_0^T u(t)dt \qquad c_n = \frac{1}{T}\int_0^T u(t)e^{-in\omega_0 t}dt$$

c_n is complex and carries the phase and amplitude information of the n^{th} component of the Fourier transform

$$c_n = \frac{a_n - ib_n}{2} \qquad d_n = \sqrt{a_n^2 + b_n^2} \qquad \phi_n = tan^{-1}\left(\frac{b_n}{a_n}\right)$$

c_n and c_{-n} are complex conjugates so that u(t) is real

$$c_n = \frac{a_n - ib_n}{2} \qquad c_{-n} = \frac{a_n + ib_n}{2}$$

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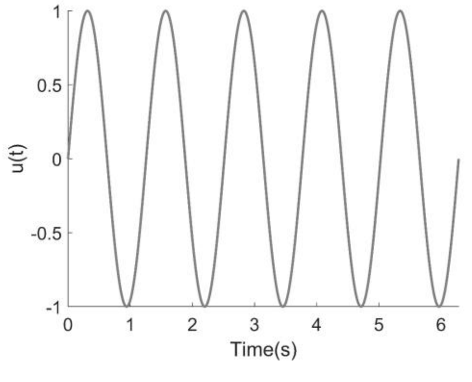
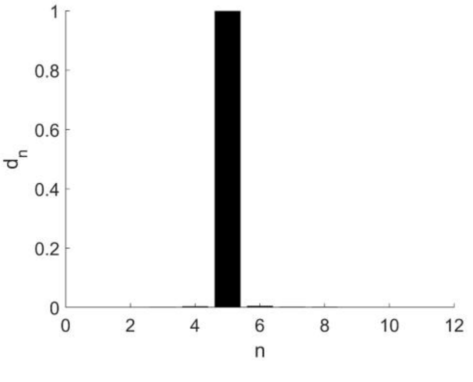
Examples

$$u(t) = sin(5\omega_0 t) \qquad \omega_0 = 1rad/s$$





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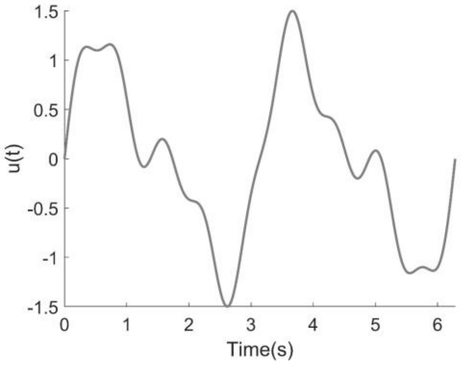
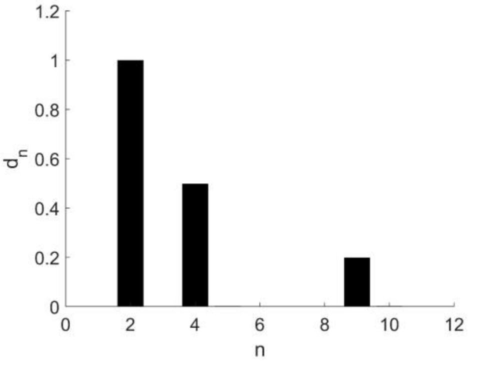
Examples

$$u(t) = sin(2\omega_0 t) + 0.5\ sin(4\omega_0 t) + 0.2\ sin(9\omega_0 t) \qquad \omega_0 = 1 rad/s$$





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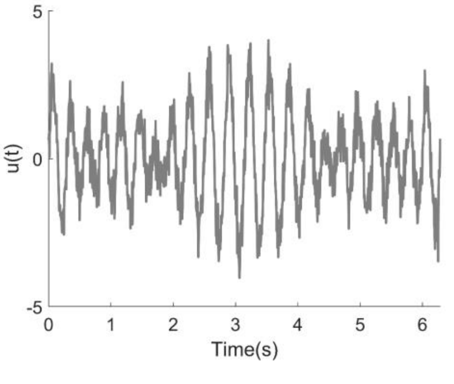
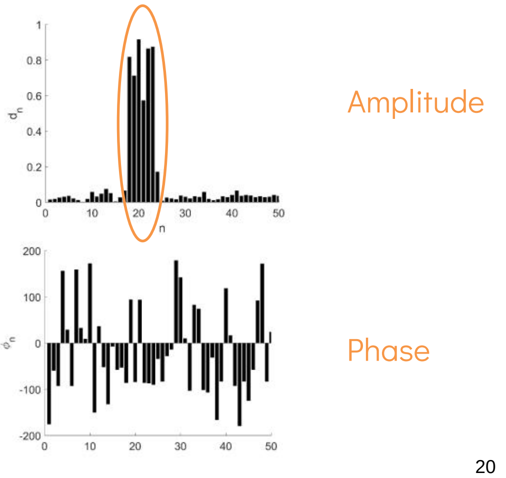
Examples

Narrow band signal with complex phases

$$\omega_0 = 1 rad/s$$





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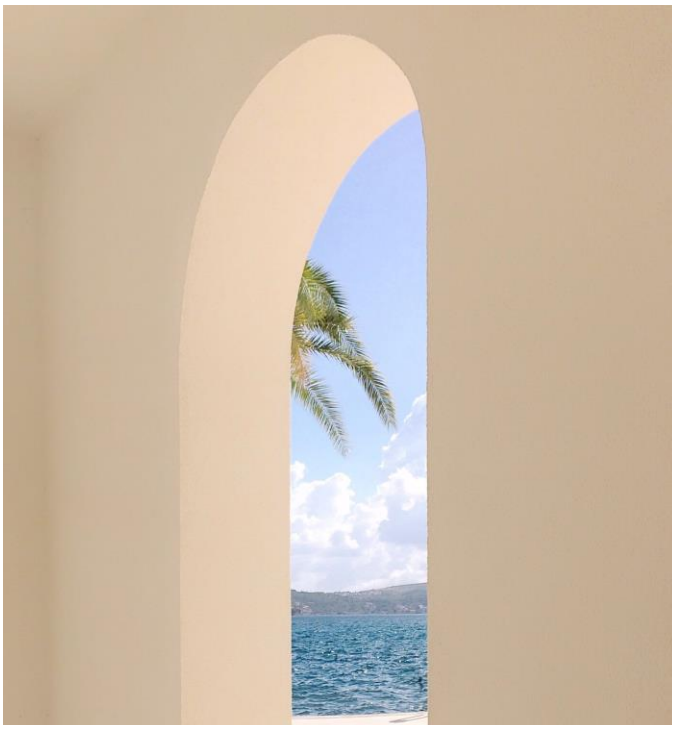
THE CONTINUOUS FOURIER TRANSFORM



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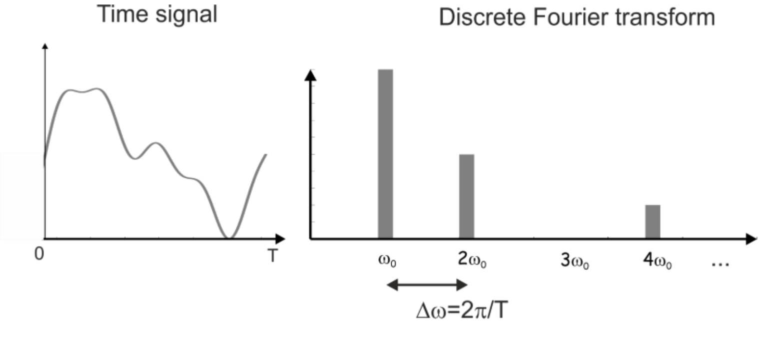
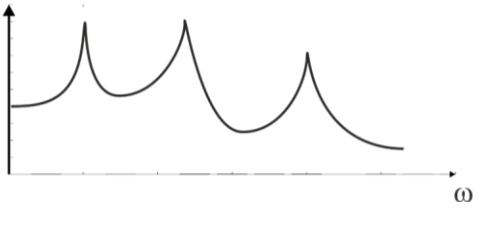
From discrete to continuous



$$T \to \infty \quad \Delta\omega \to d\omega$$



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From discrete to continuous

- Period T tends to ∞
- Discrete frequency step $\Delta\omega = \omega_0$ tends to $d\omega$
- Discrete frequency $n\omega_0$ tends to ω

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)e^{-in\omega_0 t}dt$$

$$lim_{T\to\infty} Tc_n = lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)e^{-in\omega_0 t}dt = \boxed{\int_{-\infty}^{\infty} u(t)e^{-i\omega t}dt = U(\omega)}$$

Continuous Fourier Transform of u(t)

Direct and inverse Fourier transforms

$$U(\omega) = lim_{T\to\infty} Tc_n = \int_{-\infty}^{\infty} u(t)e^{-i\omega t}dt = U(\omega)$$

$$u(t) = lim_{T\to\infty}\sum_{n=-\infty}^{n=\infty} c_n e^{in\omega_0 t} = lim_{T\to\infty}\sum_{n=-\infty}^{n=\infty} c_n \frac{T}{T} e^{in\omega_0 t}$$

$$= lim_{T\to\infty}\sum_{n=-\infty}^{n=\infty} (c_n T)\frac{\omega_0}{2\pi}e^{in\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega)e^{i\omega t}d\omega$$

$$U(\omega) = \int_{-\infty}^{\infty} u(t)e^{-i\omega t}dt$$ Continuous Fourier Transform

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega)e^{i\omega t}d\omega$$ Continuous Inverse Fourier Transform

Frequency and angular frequency

Pulsation (rad/s)

Frequency (Hz)

$$U(\omega) = \int_{-\infty}^{\infty} u(t)e^{-i\omega t}dt \longrightarrow U(f) = \int_{-\infty}^{\infty} u(t)e^{-i2\pi f t}dt$$

$$\omega = 2\pi f \Rightarrow d\omega = 2\pi df$$

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega)e^{i\omega t}d\omega \longrightarrow u(t) = \int_{-\infty}^{\infty} U(f)e^{i2\pi f t}df$$

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EXAMPLES OF CONTINUOUS FOURIER TRANSFORMS

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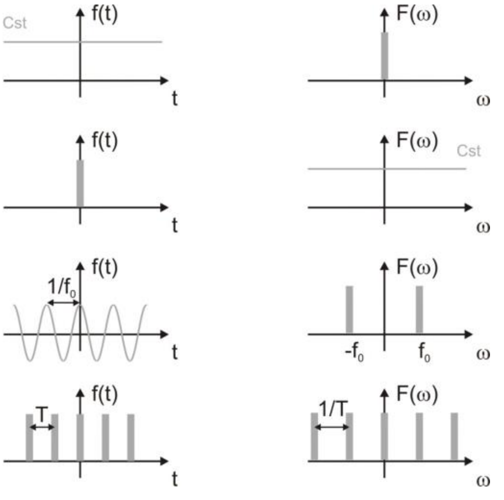
Fourier transform of basic functions

$u(t)$	$U(f)$
1	$\delta(f)$
$\delta(t)$	1
$\cos(2\pi f_0 t)$	$\frac{\delta(f-f_0)+\delta(f+f_0)}{2}$
$\sin(2\pi f_0 t)$	$\frac{\delta(f-f_0)+\delta(f+f_0)}{2i}$
$\sum_{n=-\infty}^{n=\infty} \delta(t-nT)$	$\frac{1}{T}\sum_{n=-\infty}^{n=\infty} \delta(f-\frac{n}{T})$



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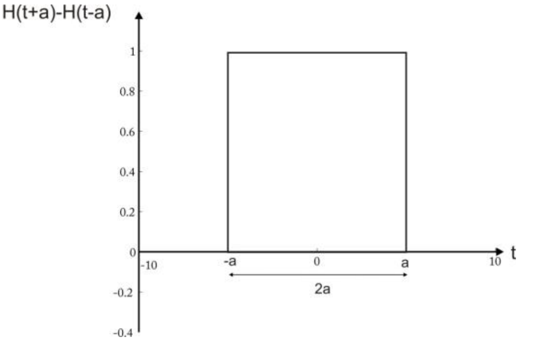
Fourier transform of an impulse

$$\boxed{u(t) = H(t+a) - H(t-a) = \begin{cases} 1 & -a < t < 0 \\ 0 & |t| > a \end{cases}}$$

$$\begin{aligned} U(f) &= \int_{-\infty}^{\infty} u(t) e^{-i2\pi f t}\, dt \\ &= \int_{-a}^{a} e^{-i2\pi f t}\, dt \\ &= \frac{-1}{i2\pi f}\left[e^{-i2\pi f t}\right]_{-a}^{a} \\ &= \frac{(e^{i2\pi f a} - e^{-i2\pi f a})}{2i\pi f} \\ &= \frac{2\sin(2\pi f a)}{2\pi f} \end{aligned}$$



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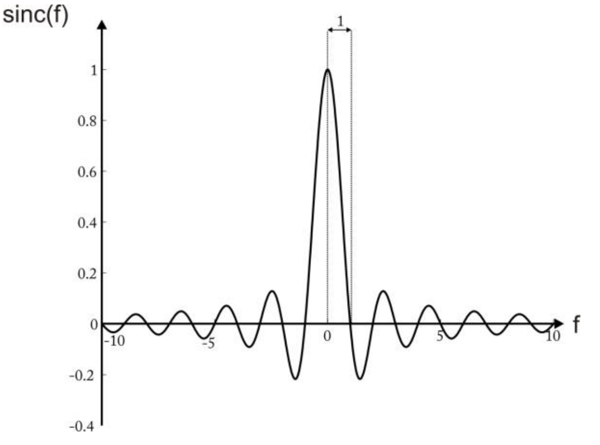
Fourier transform of an impulse

$$\frac{2\sin(2\pi fa)}{2\pi f} = 2asinc(2fa)$$

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}$$



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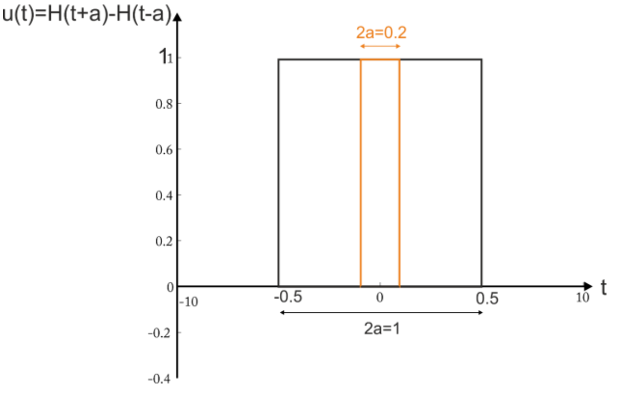
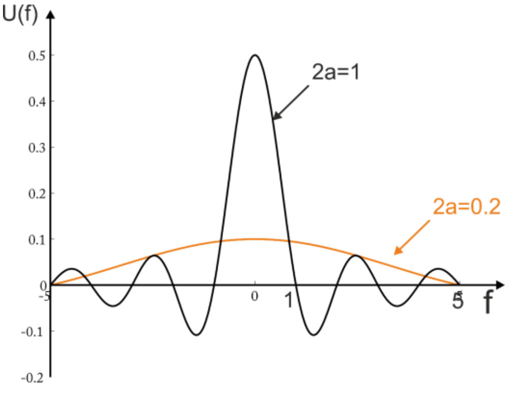
Fourier transform of an impulse





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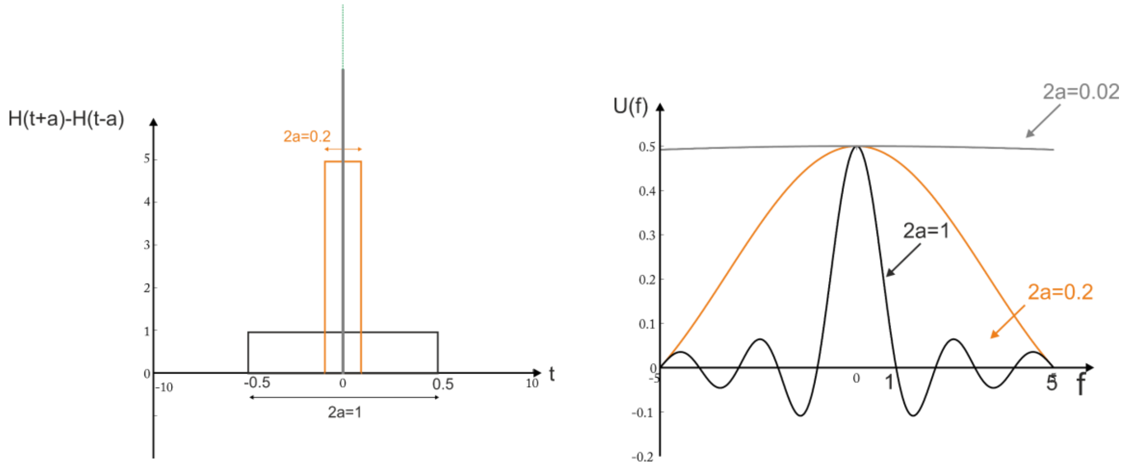
Fourier transform of an impulse



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Link between impulse response and transfer function

Duhamel's integral with harmonic excitation $f(t) = Fe^{i\omega t}$ $x(t) = Xe^{i\omega t}$

$$x(t) = Xe^{i\omega t} = \int_{-\infty}^{\infty} Fe^{i\omega\tau} h(t-\tau)d\tau = \int_{-\infty}^{\infty} Fe^{i\omega(t-\tau)} h(\tau)d\tau$$

$$= Fe^{i\omega t} \int_{-\infty}^{\infty} h(\tau)e^{-i\omega\tau} d\tau = Fe^{i\omega t} H(\omega)$$

$$\longrightarrow \quad H(\omega) = \frac{X(\omega)}{F(\omega)}$$

The Fourier transform of h(t) is the transfer function

For a single degree of freedom, we have :

$$h(t) = \frac{e^{-\xi\omega_n t}}{m\omega_d} \sin(\omega_d t) \longrightarrow \frac{X(\omega)}{F(\omega)} = \frac{1}{m}\left(\frac{1}{\omega_n^2 + 2i\xi\omega\omega_n - \omega^2}\right)$$

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The convolution theorem

Convolution in the time domain corresponds with a multiplication in the frequency domain:

$$y(t) = x(t) * h(t)$$
$$Y(f) = X(f).H(f)$$

Multiplication in the time domain corresponds to a convolution in the frequency domain

$$y(t) = x(t).h(t)$$
$$Y(f) = X(f) * H(f)$$

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FOURIER TRANSFORMS OF SAMPLED SIGNALS

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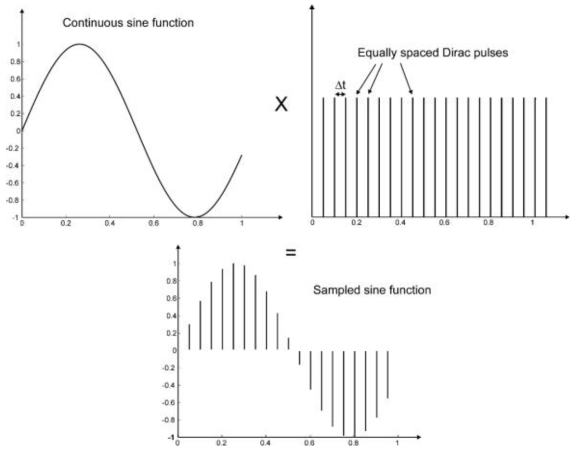
Sampling and aliasing

-In practice, vibration signals are recorded on computers at discrete time steps Δt. This is called sampling
- Sampling at time intervals Δt can be seen as multiplying the continuous function by a Dirac comb with spacing Δt



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Sampling and aliasing

Continuous Fourier Transform of a sampled signal using the convolution theorem

$$\boxed{\begin{aligned} y(t) &= x(t).h(t) \\ Y(f) &= X(f) * H(f) \end{aligned}}$$

Example :

$$x(t) = \sin(2\pi f_0 t) \rightarrow X(f) = \frac{\delta(f - f_0) + \delta(f + f_0)}{2i}$$ Continuous function

$$h(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\Delta t) \rightarrow H(f) = \frac{1}{\Delta t} \sum_{n=-\infty}^{n=\infty} \delta(f - \frac{n}{\Delta t})$$ Sampling function

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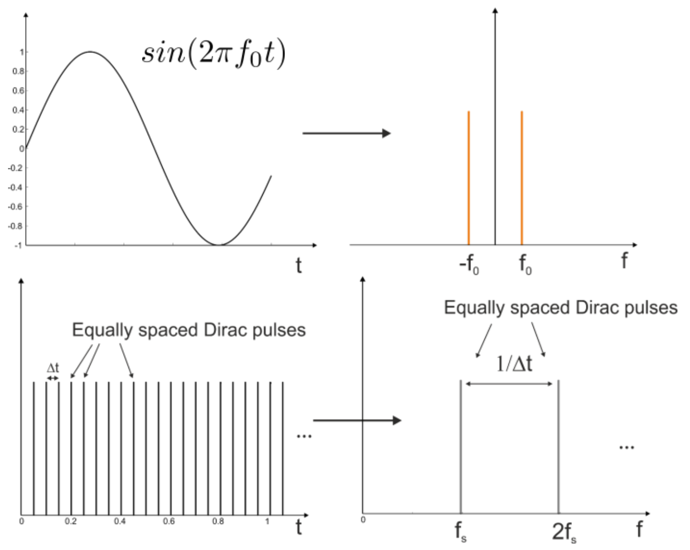
Sampling and aliasing



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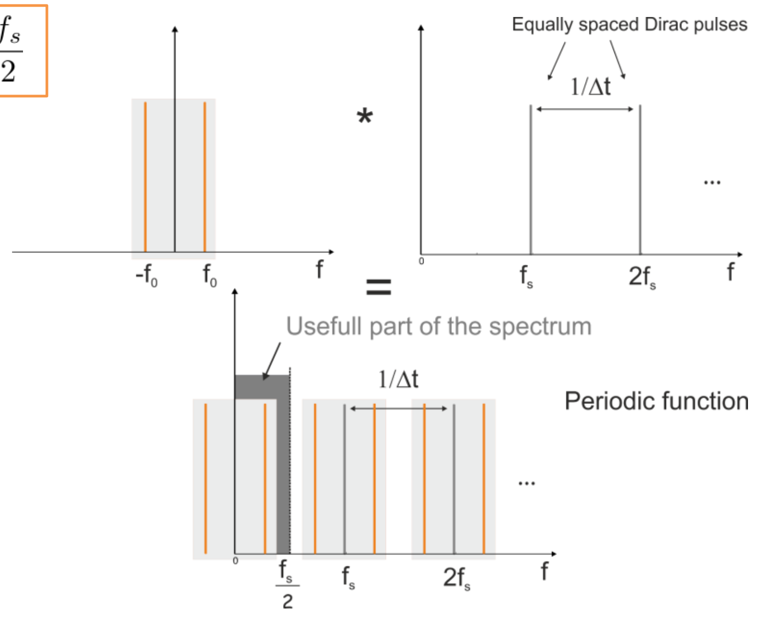
Sampling and aliasing

$$f_0 < \frac{f_s}{2}$$



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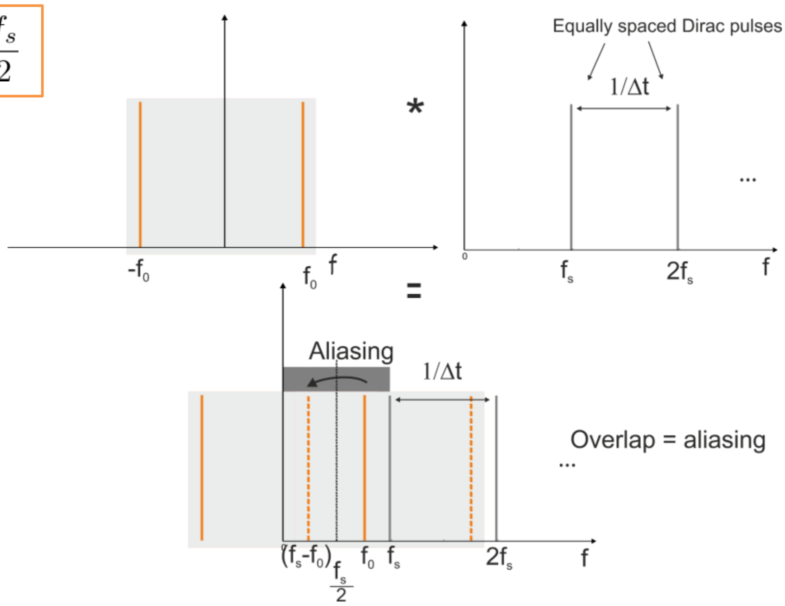
Sampling and aliasing

$$\boxed{f_0 > \frac{f_s}{2}}$$



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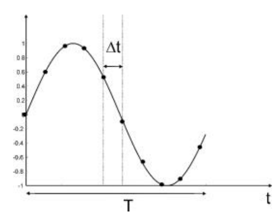
Discrete Fourier transform of sampled signals

Basic principle:

-Time interval [0 T], *N* samples, sampling time Δt



-Discrete Fourier transform computed at *N* regular frequency intervals

$$\boxed{\Delta\omega = \frac{2\pi}{N\Delta t} = \frac{2\pi}{T}}$$

The frequency resolution depends on T

resulting in a frequency band [0 f_{max}] with

$$\boxed{f_{max} = N\frac{\Delta\omega}{2\pi} = \frac{N}{T} = \frac{1}{\Delta t} = f_s}$$

and only the part below $\frac{f_s}{2}$ is useful.

The maximum frequency depends on Δt

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Discrete Fourier transform of sampled signals

Remarks:

*Taking the DFT on a time interval [0 T] can therefore be seen as implicitly assuming that the signal is periodic of period T.

* The DFT is exact only if the signal is periodic of period T, or if the signal is zero before t=0 and after t=T.

*Due to the periodicity of the Fourier transform of a sampled signal, the DFT needs to be computed only at *N* frequencies, and the useful part of the spectrum contains only *N/2* points and ranges from 0 to $f_s/2$.

*In practice, the DFT of sampled signals is computed using the FFT (Fast Fourier Transform) algorithm.

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Discrete Fourier transform in Matlab/Octave

The discrete Fourier transform is defined as :

$$c_n = \frac{1}{T}\int_0^T u(t)\, e^{-in\omega_0 t} dt$$

For a signal *u(t)* sampled at *N* regular time intervals (Δt) :

$$c_n \simeq \frac{1}{T}\sum_{j=0}^{N} \Delta t\, u(j\Delta t)\, e^{-in\omega_0 j\Delta t}$$

$$= \frac{\Delta t}{T}\underline{\sum_{j=0}^{N} u(j\Delta t)\, e^{-in\omega_0 j\Delta t}} = \frac{1}{N} fft(u)$$

Definition of FFT in Matlab/Octave

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Continuous Fourier transform in Matlab/Octave

The continuous Fourier transform is defined as :

$$U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi f t} dt = lim_{T\rightarrow\infty} T c_n$$

$$= lim_{T\rightarrow\infty} \frac{T}{N} fft(u) = (\Delta t) fft(u)$$

Converges to the continuous Fourier transform if T is large and Δt small

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PRACTICAL EXAMPLES OF DFT OF SAMPLED SIGNALS

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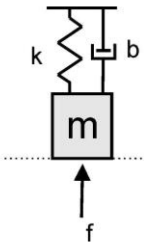
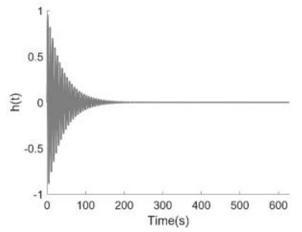
SDOF : DFT of impulse response

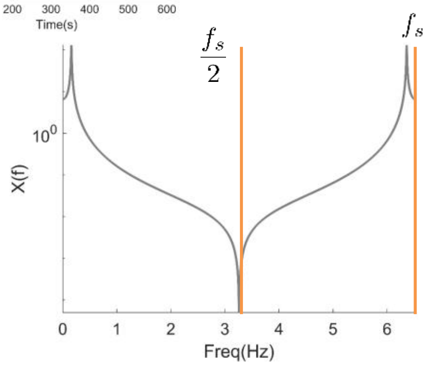
$k = 1N/m,\ m = 1kg,\ b = 0.01Ns/m$





$$\Delta t = 0.1534s \rightarrow f_s = 6.519Hz$$
$$n = 4096$$
$$T = 628s \rightarrow \Delta f = 0.001592Hz$$

Approximation of the transfer function in the frequency band [0 $f_s/2$]



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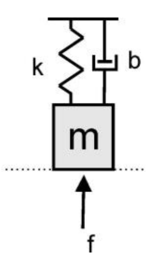
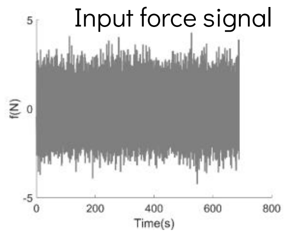
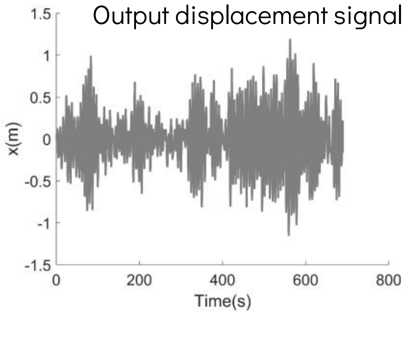
SDOF : response to random excitation

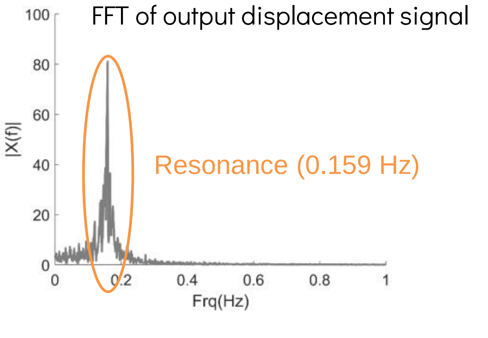
$k = 1N/m,\ m = 1kg,\ b = 0.01Ns/m$





$$\Delta t = 0.01534s \rightarrow f_s = 65.19Hz$$
$$n = 45056$$
$$T = 691s \rightarrow \Delta f = 0.00144Hz$$





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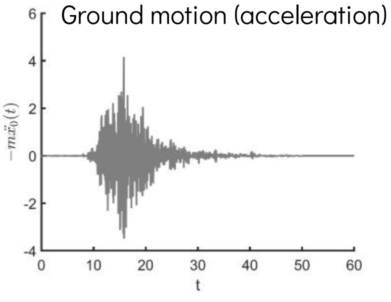
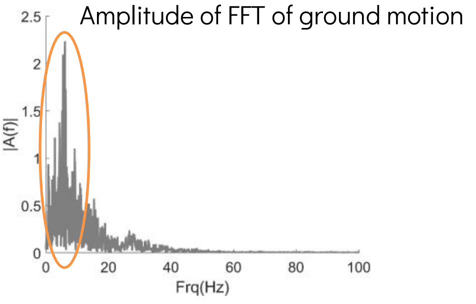
Frequency analysis of Santa Cruz Earthquake

Santa Cruz 1989 Earthquake

$$\Delta t = 0.005s \rightarrow f_s = 200Hz$$
$$n = 10000$$
$$T = 50s \rightarrow \Delta f = 0.02Hz$$





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Frequency analysis of engine vibrations

https://www.youtube.com/watch?v=3Z1KjwIB_-c

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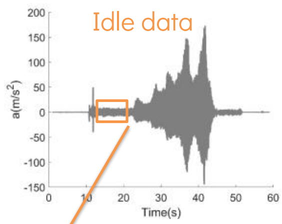
Frequency analysis of engine vibrations

[mide.com]

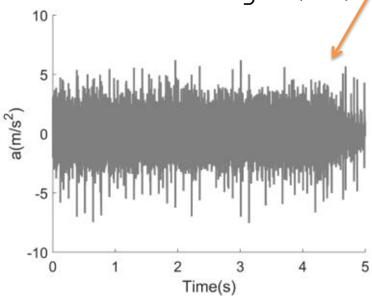


$$\Delta t = 50\mu s \rightarrow f_s = 20kHz$$

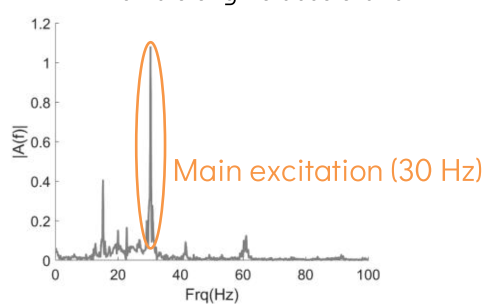
$$n = 100001$$

$$T = 5s \rightarrow \Delta f = 0.2Hz$$

Acceleration signal (idle)



FFT of idle engine acceleration



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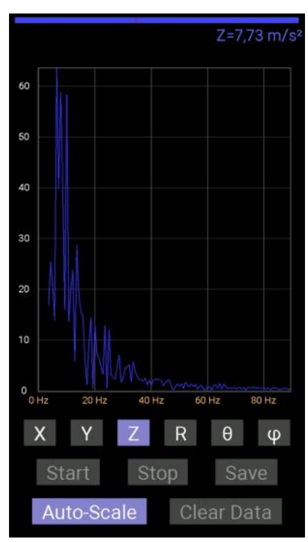
Determining the natural frequency of an object with your phone



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