CONTINUOUS SYSTEMS

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CONTINUOUS SYSTEMS IN REAL LIFE

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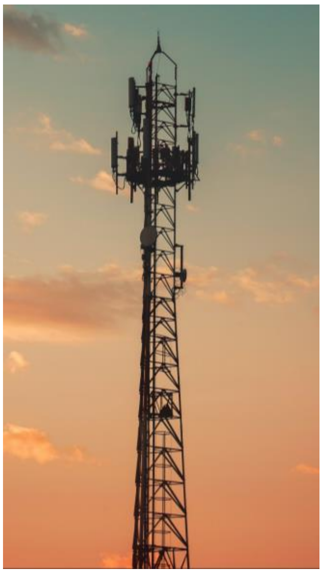
Simple continuous systems : civil engineering



Beam –
Bar kinematics

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Simple continuous systems : mechanical engineering

Beam –
Bar kinematics

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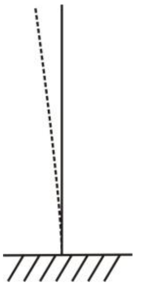
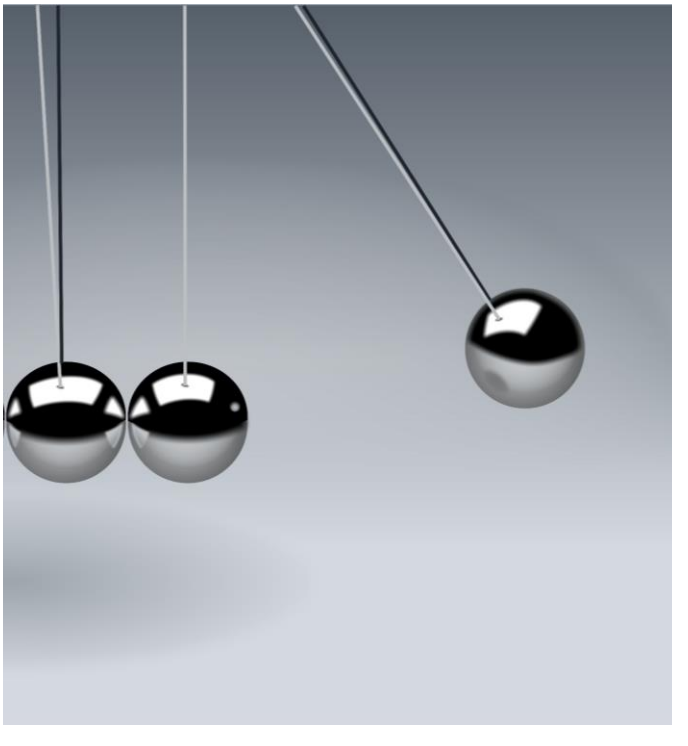
Equivalent continuous systems



Cantilever beam

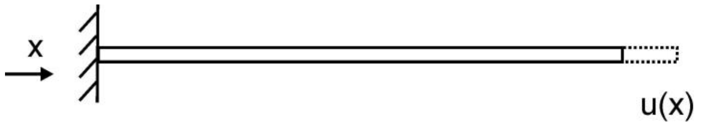
Simply supported beam

EQUATIONS OF MOTION FOR BEAMS AND BARS



Boundary conditions for bars

For bars:



$$N = EA\varepsilon$$ Normal force

$$\varepsilon = \frac{du}{dx} = u'$$ Longitudinal strain

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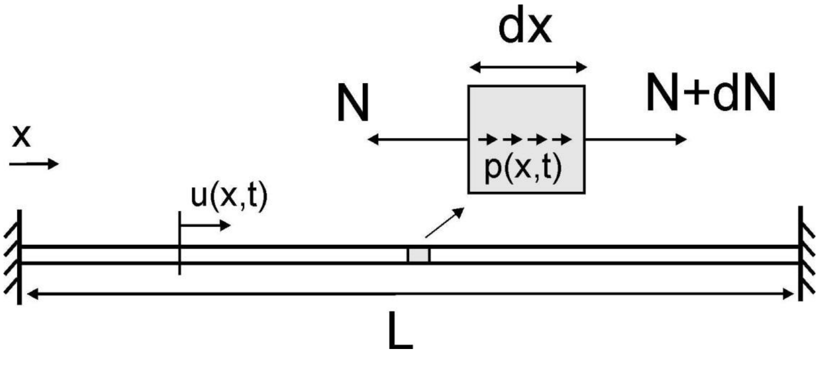
Equation of motion for bars



- axial displacement u(x,t)
- can be seen as an infinite number of small mass-spring systems in series
  -> Infinite number of eigenfrequencies and mode shapes

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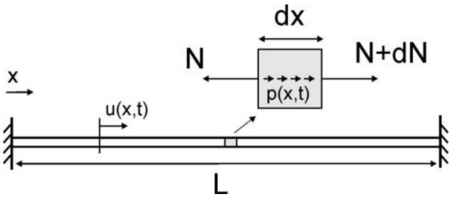
Equation of motion for bars



Equilibrium :

$$-N + p(x,t)\,dx + N + dN = (\rho A \ddot{u}(x,t))\,dx$$

p(x,t) = axial load per unit length on the bar
A = surface of the section
ρ = density

$$\frac{dN}{dx} = \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = EA\frac{\partial^2 u(x,t)}{\partial x^2} = \rho A \ddot{u}(x,t) - p(x,t)$$

$$\boxed{EA\frac{\partial^2 u(x,t)}{\partial x^2} - \rho A \ddot{u}(x,t) = -p(x,t)}$$

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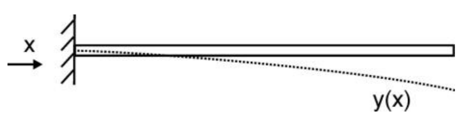
Boundary conditions for beams

For beams:

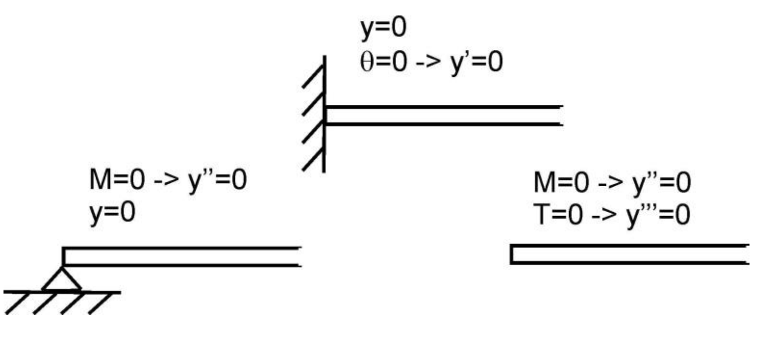


$M = -EI\chi$ Bending moment

$\theta = \frac{dy}{dx} = y'$ Rotation

$\chi = y''$ Curvature

$T = \frac{dM}{dx} = -EIy'''$ Shear force



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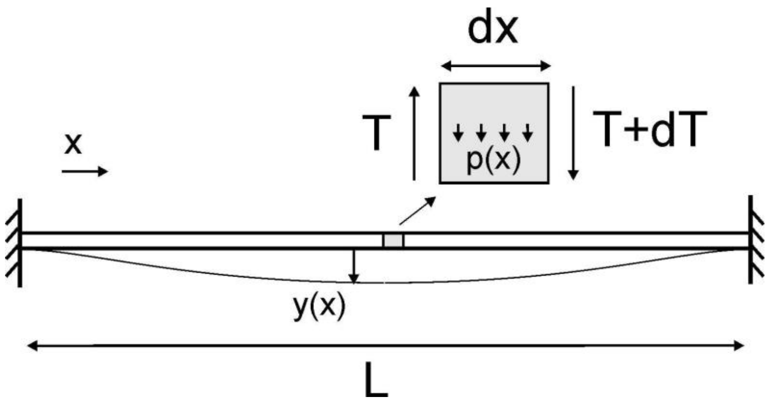
Equation of motion for beams



- transversal displacement of the neutral axis y(x,t)

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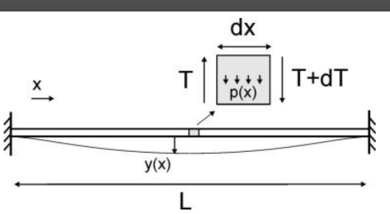
Equation of motion for beams



Equilibrium : $-T + p(x,t)\,dx + T + dT = (\rho A \ddot{y})\,dx$

p(x,t) = vertical load per unit length on the beam
A = surface of the section
ρ = density

$$\frac{dT}{dx} = -EI\frac{d^4y}{dx^4} = \rho A\ddot{y} - p(x,t)$$

$$\boxed{EI\frac{d^4y}{dx^4} + \rho A\ddot{y} = p(x,t)}$$

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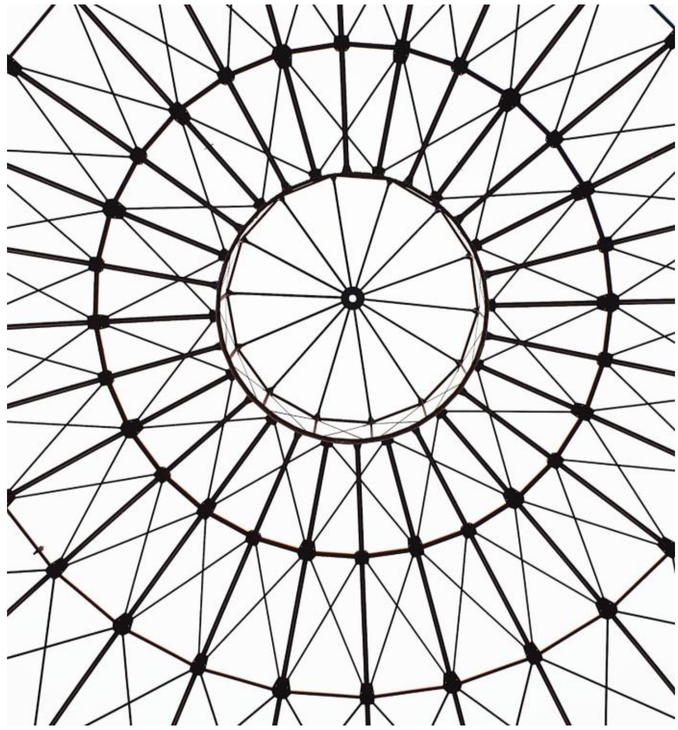
MODE SHAPES FOR BARS



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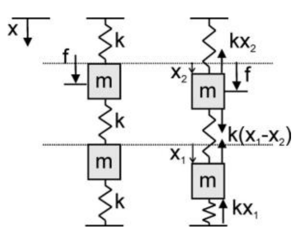
Eigenfrequencies

N degrees of freedom (dofs) system = n eigenfrequencies



Continuous system = infinite number of DOFs = infinite number of eigenfrequencies

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Mode shapes and eigenfrequencies for bars

$$\boxed{EA\frac{\partial^2 u(x,t)}{\partial x^2} - \rho A\ddot{u}(x,t) = -p(x,t)}$$

General solution (p(x,t)=0) : $u(x,t) = U(x)e^{i\omega t}$

$$EA\frac{d^2U}{dx^2} + \rho A\omega^2 U = 0$$

$$\frac{d^2U}{dx^2} + \frac{\rho}{E}\omega^2 U = 0$$

Characteristic equation: $U(x) = Ae^{rx}$! The variable is x not t

$$r^2 + \frac{\rho}{E}\omega^2 = 0 \longrightarrow r_{1,2} = \pm i\omega\sqrt{\frac{\rho}{E}}$$

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Mode shapes and eigenfrequencies for bars

General solution

$$U(x) = Acos(\omega\sqrt{\frac{\rho}{E}}x) + Bsin(\omega\sqrt{\frac{\rho}{E}}x)$$

A and B depend on the boundary conditions

Example : Bar fixed at x=0 and x=L

$$U(0) = 0 \Rightarrow A = 0$$

$$U(L) = Bsin(\omega\sqrt{\frac{\rho}{E}}L) = 0 \longrightarrow \omega\sqrt{\frac{\rho}{E}}L = n\pi \qquad n = 1, ..., \infty$$

Eigenfrequencies	Mode shapes	
$\omega_n = n\frac{\pi}{L}\sqrt{\frac{E}{\rho}}$	$U(x)_n = sin(\frac{n\pi x}{L})$	$n = 1, ..., \infty$

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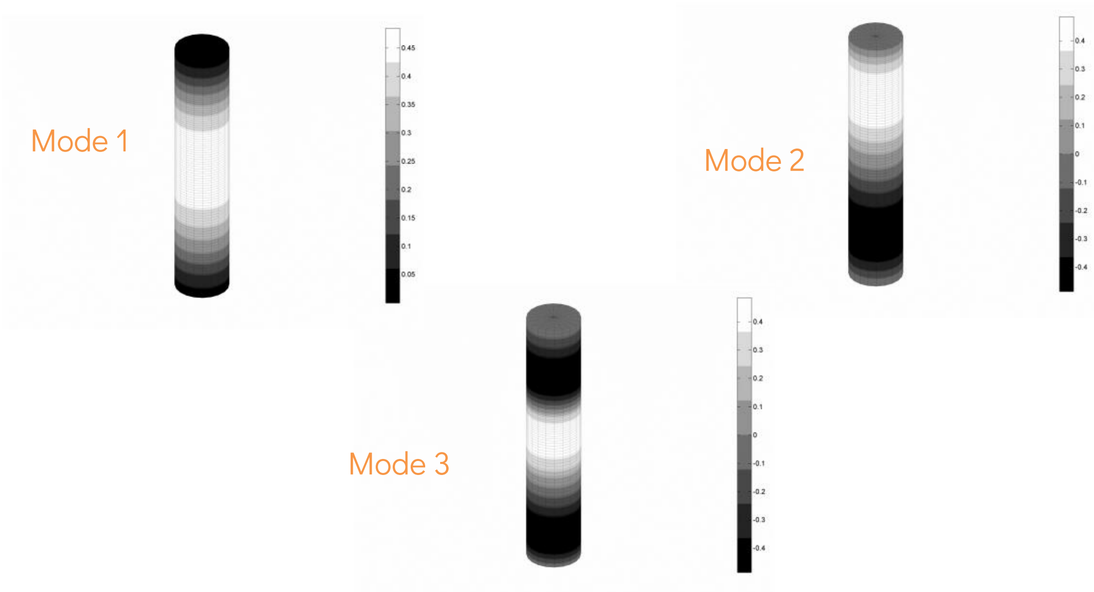
Mode shapes and eigenfrequencies for bars



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MODE SHAPES FOR BEAMS

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Mode shapes and eigenfrequencies for beams

$$\boxed{EI\frac{\partial^4 y(x,t)}{\partial x^4} + \rho A\ddot{y}(x,t) = p(x,t)}$$

General solution (p(x,t)=0) : $y(x,t) = Y(x)e^{i\omega t}$

$$EI\frac{d^4Y}{dx^4} - \rho A\omega^2 Y = 0$$

$$\frac{d^4Y}{dx^4} - \xi^4 Y = 0 \qquad \xi^4 = \frac{\rho A}{EI}\omega^2$$

Characteristic equation: $Y(x) = Ae^{rx}$! The variable is x not t

$$r^4 - \xi^4 = 0 \longrightarrow \begin{matrix} r_{1,2} = \pm i\xi \\ r_{3,4} = \pm \xi \end{matrix}$$

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Mode shapes and eigenfrequencies for bars

General solution

$$Y(x) = Acos(\xi x) + Bsin(\xi x) + Ccosh(\xi x) + Dsinh(\xi x)$$

A, B, C and D depend on the boundary conditions

Example : Simply supported beam

$$Y(0) = Y''(0) = Y(L) = Y''(L) = 0$$

$$sin(\xi L)sinh(\xi L) = 0 \longrightarrow \xi L = n\pi \qquad n = 1, ..., \infty$$

Eigenfrequencies

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}}$$

Mode shapes

$$Y_n(x) = sin\frac{n\pi x}{L}$$

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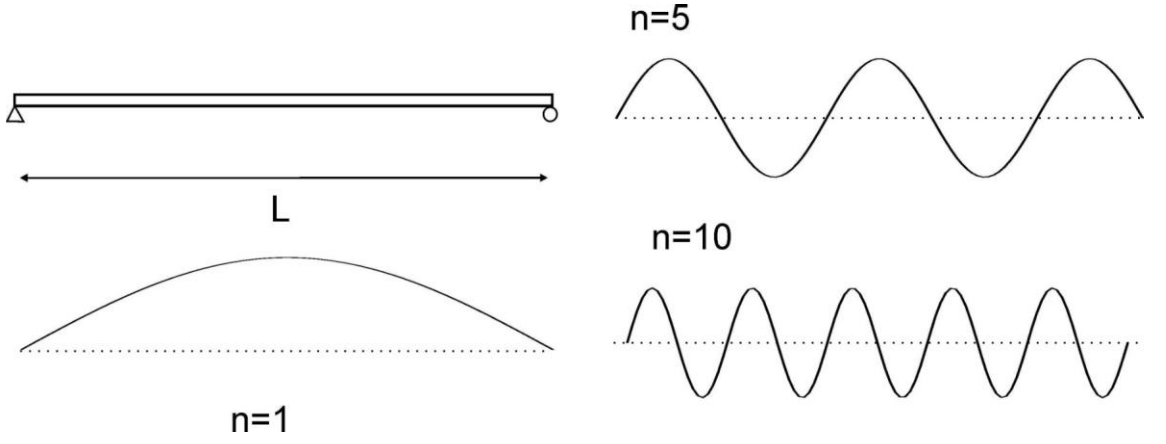
Mode shapes and eigenfrequencies for beams

Example 1: Simply supported beam



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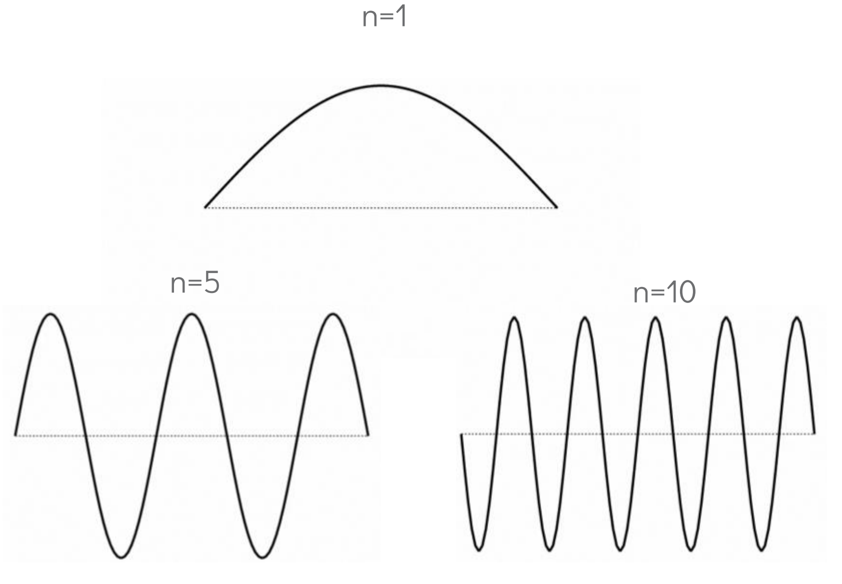
Mode shapes and eigenfrequencies for beams



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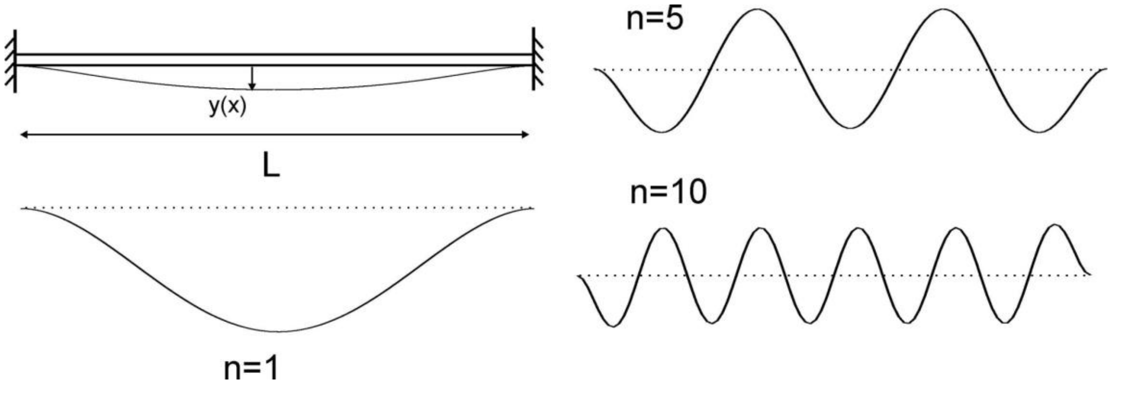
Mode shapes and eigenfrequencies for beams

Example 2: Double cantilever beam



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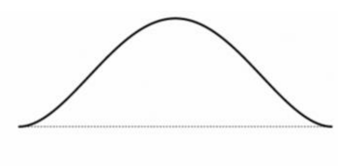
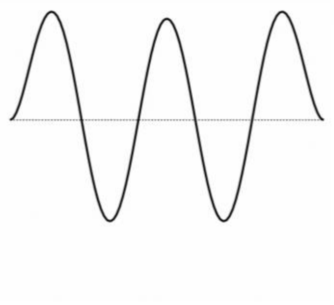
Mode shapes and eigenfrequencies for beams

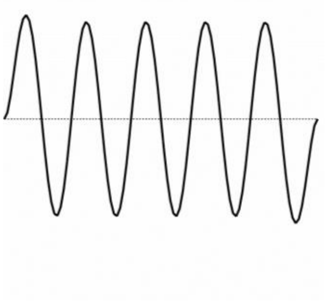
n=1



n=5



n=10



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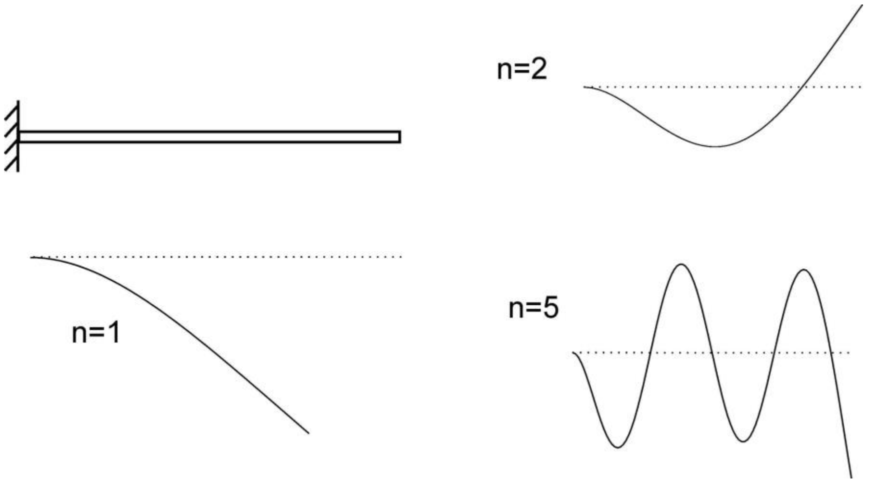
Mode shapes and eigenfrequencies for beams

Example 3: Cantilever beam



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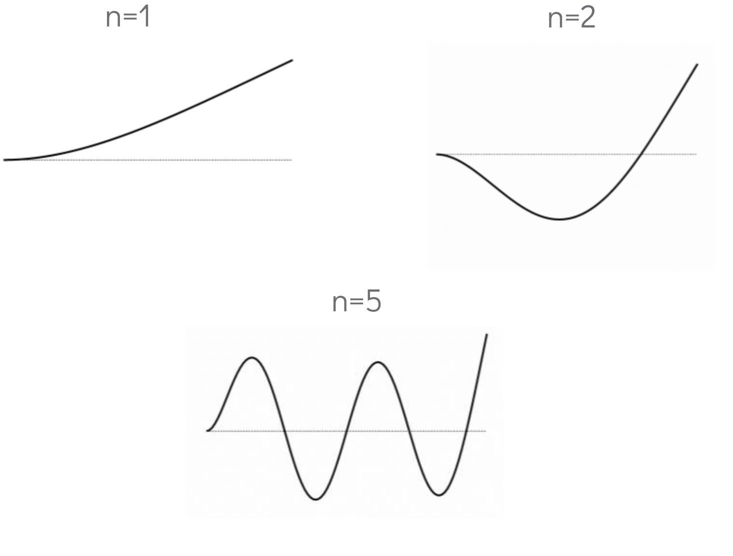
Mode shapes and eigenfrequencies for beams



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Mode shapes and eigenfrequencies for beams

https://youtu.be/1Z-d_DlxVSQ 27

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PROJECTION IN THE MODAL BASIS

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Projection in the modal basis for bars

Orthogonality :

$$\boxed{\begin{aligned}\int_0^L \rho A U_i U_j \, dx &= \delta_{ij}\mu_i \\ \int_0^L EAU_i' U_j' \, dx &= \delta_{ij}\mu_i\omega_i^2\end{aligned}}$$

$$EA\frac{\partial^2 u(x,t)}{\partial x^2} - \rho A\frac{\partial^2 u(x,t)}{\partial t^2} = -p(x,t) \qquad u(x,t) = \sum_{j=1}^{\infty} U_j(x)z_j(t)$$

$$EA\sum_{j=1}^{\infty} U_j'' z_j - \rho A\sum_{j=1}^{\infty} U_j \ddot{z}_j = -p(x,t)$$

$$\int_0^L \left(EA\sum_{j=1}^{\infty} U_j'' U_i\right) z_j \, dx - \int_0^L \left(\rho A\sum_{j=1}^{\infty} U_j U_i\right) \ddot{z}_j \, dx = \int_0^L -p(x,t)U_i \, dx$$

$$\boxed{\mu_i \ddot{z}_i + \mu_i\omega_i^2 z_i = F_i} \qquad F_i = \int_0^L p(x,t)U_i \, dx$$

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Projection in the modal basis

$$u(x,t) = \sum_{j=1}^{\infty} U_j(x)z_j(t)$$

The solution can be obtained by solving an <u>infinite</u> set of independent equations of the type

$$\boxed{\mu_i \ddot{z}_i + \mu_i\omega_i^2 z_i = F_i}$$

This equation corresponds to the equation of motion of a sdof system with

- mass μ_i (modal mass)
- stiffness $\mu_i\omega_i^2$
- angular eigenfrequency ω_i
- excitation F_i (modal excitation)

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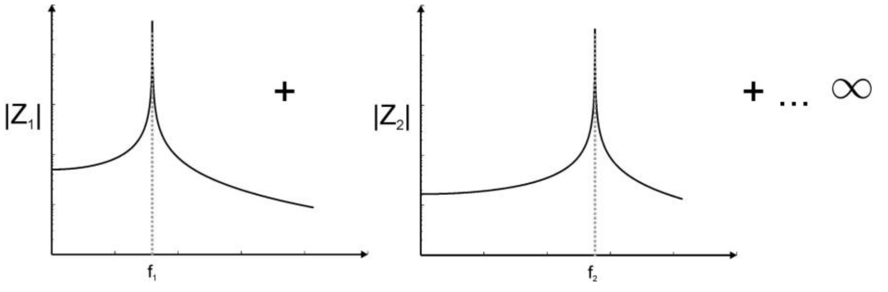
Projection in the modal basis

The solution is the sum of sdof oscillators :

$$U(x) = \sum_{i=1}^{\infty} \frac{\int_0^L P(x)U_i(x)dx}{\mu_i(\omega_i^2 - \omega^2)} U_i(x)$$



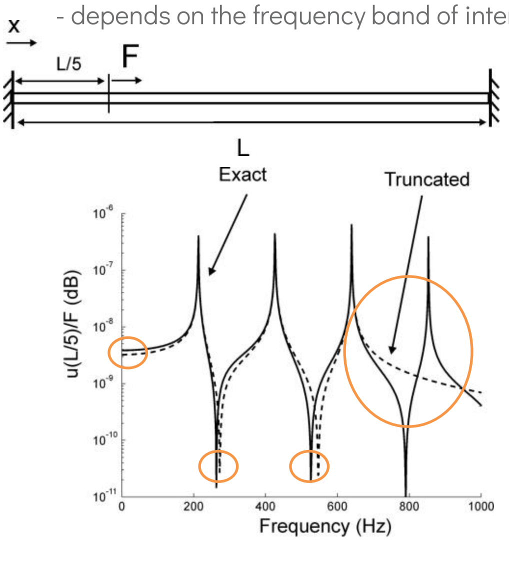
In practice, truncation is needed ...

$$\rightarrow \quad U(x) = \sum_{i=1}^{n} \frac{\int_0^L P(x)U_i(x)dx}{\mu_i(\omega_i^2 - \omega^2)} U_i(x)$$

Truncated modal basis

Rules for truncation :

- depends on the frequency band of excitation
- depends on the frequency band of interest for the response



In practice :

$$\omega < \frac{\omega_n}{1.5}$$

ω_n is the last eigenfrequency used in the truncated sum

The approximation can be improved using static correction

Projection in the modal basis for beams

Orthogonality :

$$\boxed{\begin{aligned}\int_0^L \rho A Y_i Y_j \, dx = \delta_{ij}\mu_i \\ \int_0^L EI Y_i'' Y_j'' \, dx = \delta_{ij}\mu_i\omega_i^2\end{aligned}}$$

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \rho A \frac{\partial^2 y(x,t)}{\partial t^2} = p(x,t) \qquad y(x,t) = \sum_{j=1}^{\infty} Y_j(x) z_j(t)$$

$$EI\sum_{j=1}^{\infty} Y_j^{IV} z_j + \rho A \sum_{j=1}^{\infty} Y_j \ddot{z}_j = p(x,t)$$

$$\int_0^L \left(EI \sum_{j=1}^{\infty} Y_j^{IV} Y_i \right) z_j \, dx + \int_0^L \left(\rho A \sum_{j=1}^{\infty} Y_j Y_i \right) \ddot{z}_j \, dx = \int_0^L p(x,t) Y_i \, dx$$

$$\boxed{\mu_i \ddot{z}_i + \mu_i \omega_i^2 z_i = F_i} \qquad F_i = \int_0^L p(x,t) Y_i \, dx$$

Discrete vs continuous systems

MDOF	BARS	BEAMS

Orthogonality conditions

$$\boxed{\begin{aligned}\psi_i^T M \psi_j &= \delta_{ij}\mu_i \\ \psi_i^T K \psi_j &= \delta_{ij}\mu_i\omega_i^2\end{aligned}} \qquad \boxed{\begin{aligned}\int_0^L \rho A U_i U_j \, dx = \delta_{ij}\mu_i \\ \int_0^L EA U_i' U_j' \, dx = \delta_{ij}\mu_i\omega_i^2\end{aligned}} \qquad \boxed{\begin{aligned}\int_0^L \rho A Y_i Y_j \, dx = \delta_{ij}\mu_i \\ \int_0^L EI Y_i'' Y_j'' \, dx = \delta_{ij}\mu_i\omega_i^2\end{aligned}}$$

Projection in the modal basis

$$\boxed{\mu_i \ddot{z}_i + \mu_i \omega_i^2 z_i = F_i}$$

$$F_i = \psi_i^T F \qquad F_i = \int_0^L p(x,t) U_i \, dx \qquad F_i = \int_0^L p(x,t) Y_i \, dx$$

Damping models

For proportional (global) damping models

$$\boxed{\mu_i \ddot{z}_i + 2\xi_i \mu_i \omega_i \dot{z}_i + \mu_i \omega_i^2 z_i = F_i}$$

Rayleigh damping $\Rightarrow \xi_i = \frac{1}{2}\left(\alpha\omega_i + \frac{\beta}{\omega_i}\right)$

Loss factor $\Rightarrow \xi_i = \frac{\eta}{2}$ Constant modal damping

Modal damping $\Rightarrow \xi_i$ For each mode