REDUCTION TO SDOF SYSTEMS

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EQUIVALENT MASS AND STIFFNESS APPROACH

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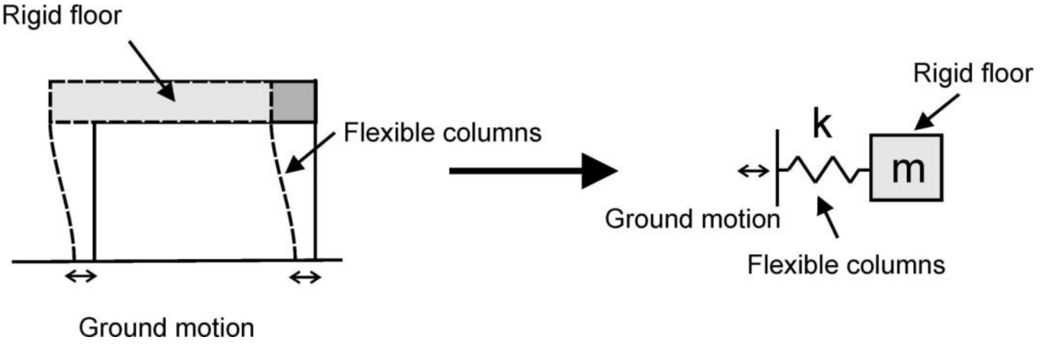
Portal frame



-Equivalent stiffness?
-Equivalent mass ?
-Validity (frequency band) ?

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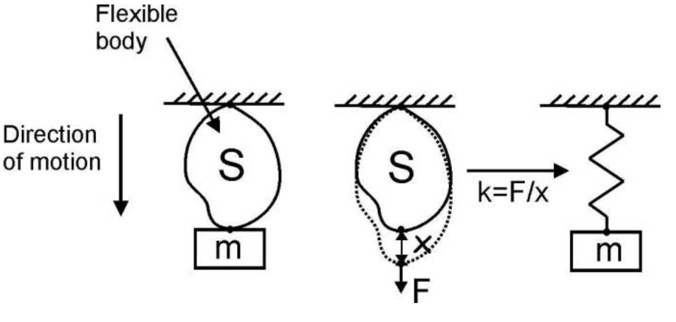
Equivalent stiffness computation

General principle :



F in the direction of excitation
(and motion) of the mass

Computation : - simplified model
- numerical approximation (finite elements)

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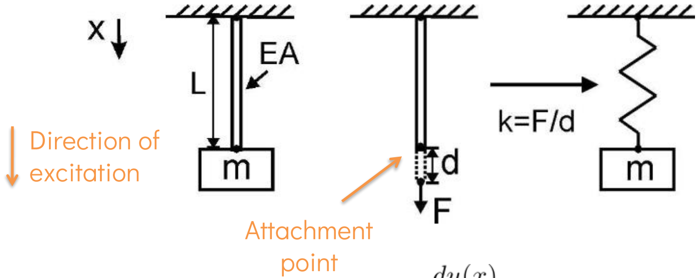
Equivalent stiffness : bar in traction



Constitutive equation: $\sigma_x = E\varepsilon_x = E\dfrac{du(x)}{dx}$ Stress is constant

$$N = \int_S \sigma_x = EA\frac{du}{dx}$$

Static equilibrium

$$N = EA\frac{du}{dx} = F \longrightarrow u(x) = \frac{F}{EA}x$$

Displacement of the attachment point (in the direction of excitation): $d = \dfrac{F}{EA}L \longrightarrow \boxed{k = \dfrac{EA}{L}}$

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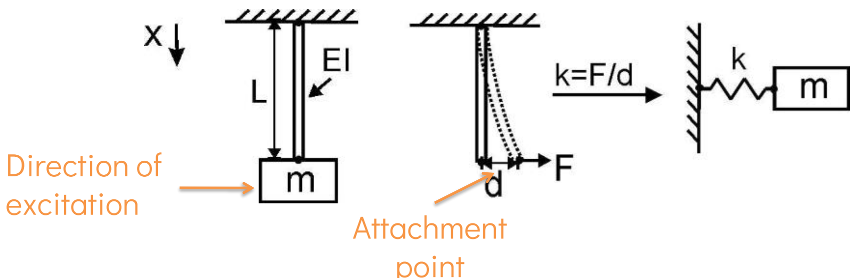
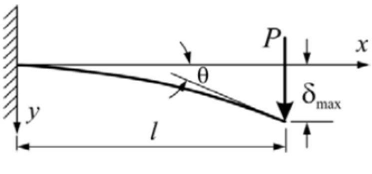
Equivalent stiffness : bending cantilever beam



From tables



$$\theta = \frac{Pl^2}{2EI} \qquad y = \frac{Px^2}{6EI}(3l - x)$$

$$\delta_{max} = \frac{Pl^3}{3EI} \longrightarrow \boxed{k = \frac{3EI}{L^3}}$$

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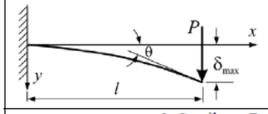
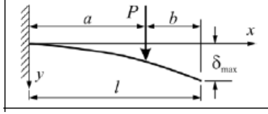
Equivalent stiffness of beams

Equivalent stiffness of beams in bending can be computed using formulas from resistance of materials or using tables :

BEAM TYPE	SLOPE AT FREE END	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM DEFLECTION
1. Cantilever Beam – Concentrated load P at the free end			
	$\theta = \frac{Pl^2}{2EI}$	$y = \frac{Px^2}{6EI}(3l - x)$	$\delta_{max} = \frac{Pl^3}{3EI}$
2. Cantilever Beam – Concentrated load P at any point			
	$\theta = \frac{Pa^2}{2EI}$	$y = \frac{Px^2}{6EI}(3a - x)$ for $0 < x < a$ $y = \frac{Pa^2}{6EI}(3x - a)$ for $a < x < l$	$\delta_{max} = \frac{Pa^2}{6EI}(3l - a)$

http://home.eng.iastate.edu/~shermanp/STAT447/STAT%20Articles/Beam_Deflection_Formulae.pdf

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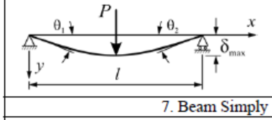
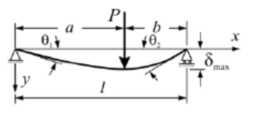
Equivalent stiffness of beams

BEAM TYPE	SLOPE AT ENDS	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM AND CENTER DEFLECTION
6. Beam Simply Supported at Ends – Concentrated load P at the center			
	$\theta_1 = \theta_2 = \frac{Pl^2}{16EI}$	$y = \frac{Px}{12EI}\left(\frac{3l^2}{4} - x^2\right)$ for $0 < x < \frac{l}{2}$	$\delta_{max} = \frac{Pl^3}{48EI}$
7. Beam Simply Supported at Ends – Concentrated load P at any point			
	$\theta_1 = \frac{Pb(l^2 - b^2)}{6lEI}$ $\theta_2 = \frac{Pab(2l - b)}{6lEI}$	$y = \frac{Pbx}{6lEI}\left(l^2 - x^2 - b^2\right)$ for $0 < x < a$ $y = \frac{Pb}{6lEI}\left[\frac{l}{b}(x - a)^3 + \left(l^2 - b^2\right)x - x^3\right]$ for $a < x < l$	$\delta_{max} = \frac{Pb\left(l^2 - b^2\right)^{3/2}}{9\sqrt{3}\,lEI}$ at $x = \sqrt{\left(l^2 - b^2\right)/3}$ $\delta = \frac{Pb}{48EI}\left(3l^2 - 4b^2\right)$ at the center, if $a > b$

http://home.eng.iastate.edu/~shermanp/STAT447/STAT%20Articles/Beam_Deflection_Formulae.pdf

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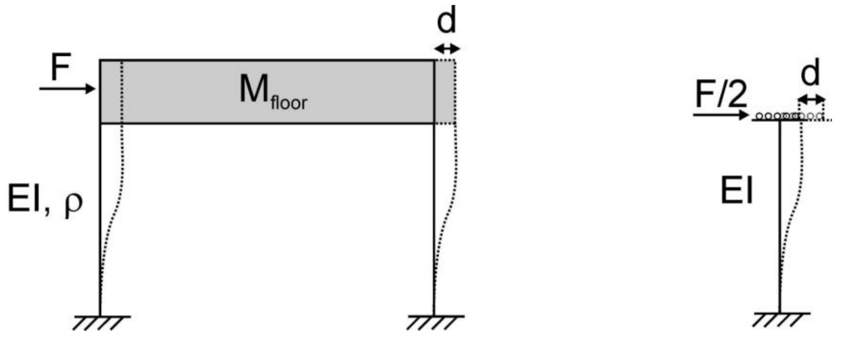
Portal frame example



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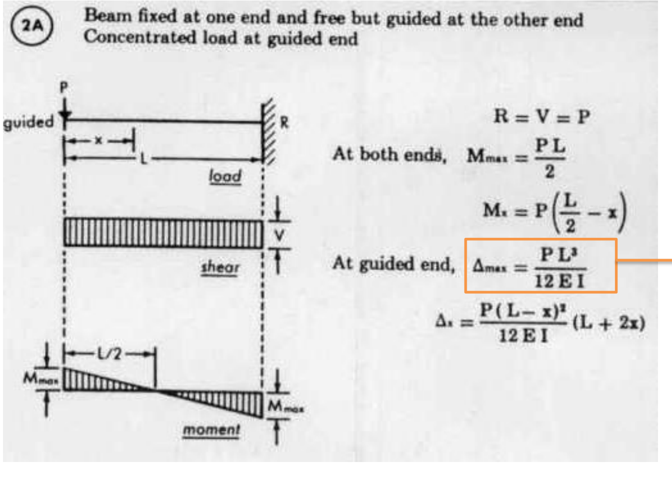
Portal frame example



$$k = 24\frac{EI}{L^3}$$

$$m = M_{floor}$$

$$\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{24EI}{L^3 M_{floor}}}(Hz)}$$

https://www.sprecace.com/node/43

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Equivalent mass : energy method

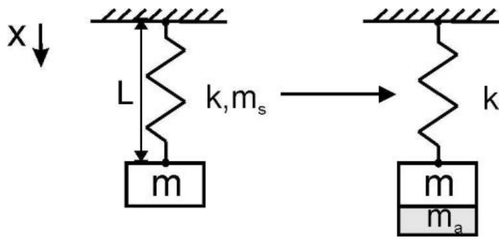
Kinetic energy of a mass-spring system

$$E_k = \frac{1}{2}mv^2$$

Example 1 :

$$v(x) = v(L)\frac{x}{L}$$

$$\rho = m_s/L$$



Kinetic energy of the spring

$$E_k = \frac{1}{2}\int_0^L \frac{m_s}{L} v(L)^2 \frac{x^2}{L^2}dx = \frac{1}{2}\frac{m_s}{3}v(L)^2$$ Additional mass

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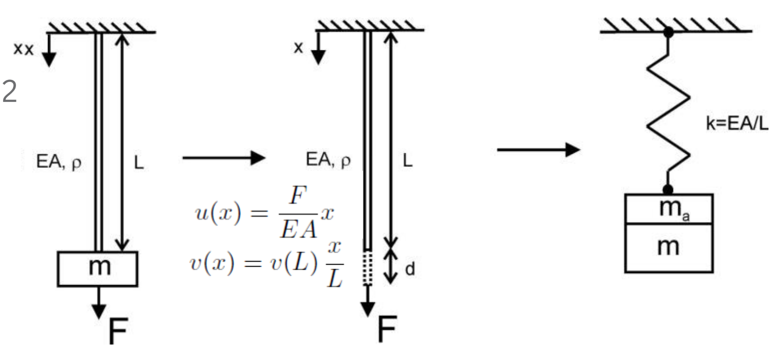
Equivalent mass : energy method

Kinetic energy of a mass-spring system $E_k = \frac{1}{2}mv^2$

Example 2



Kinetic energy of the bar

$$E_k = \frac{1}{2}\int_0^L \rho A v(L)^2 \frac{x^2}{L^2}dx = \frac{1}{2}\frac{\rho A L}{3}v(L)^2$$ Additional mass m_a

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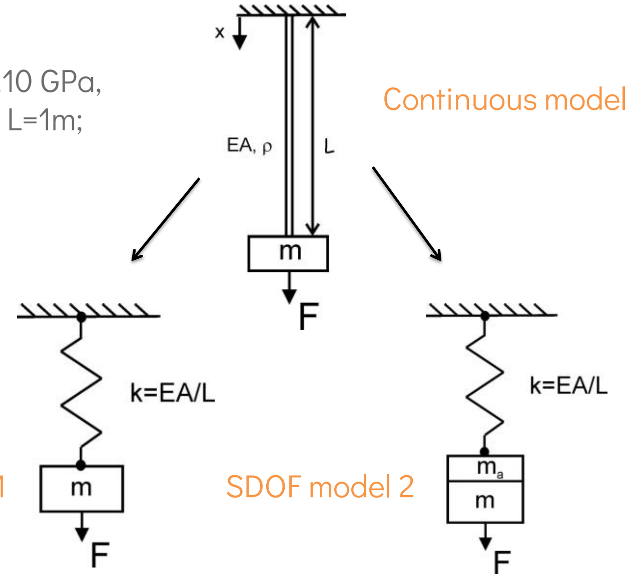
Limits of the approximation

m=100 kg, E =210 GPa,
ρ=7800 kg/m^3, L=1m;
A = 0.0004 m^2



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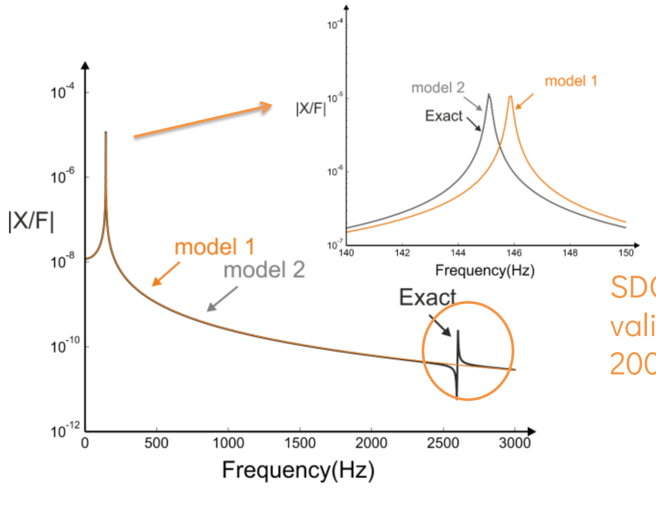
Limits of the approximation

k =EA/L = 8.4 10^7 N/m → f_n=145.87 Hz (mass M)
M = 100 kg

m_a = ρAL/3 = 1.04 kg <<M → f_n=145.11 Hz (mass M+m_a)



SDOF hypothesis not valid for frequencies > 2000 Hz

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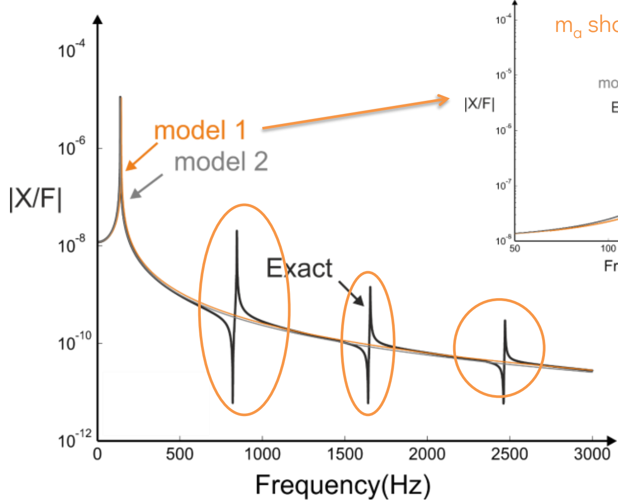
Limits of the approximation

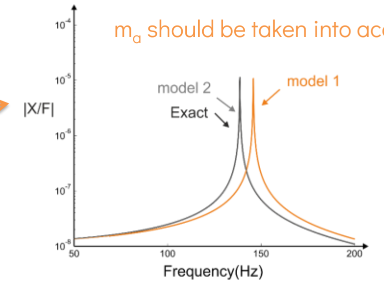
$k = EA/L = 8.4\ 10^7$ N/m → f_n=145.87 Hz (mass M)
M = 100 kg

Multiply mass of bar by 10

$m_a = \rho AL/3 = 10.4$ kg <M → f_n=138.83 Hz (mass M+m_a)



m_a should be taken into account



SDOF hypothesis not valid for frequencies > 500 Hz

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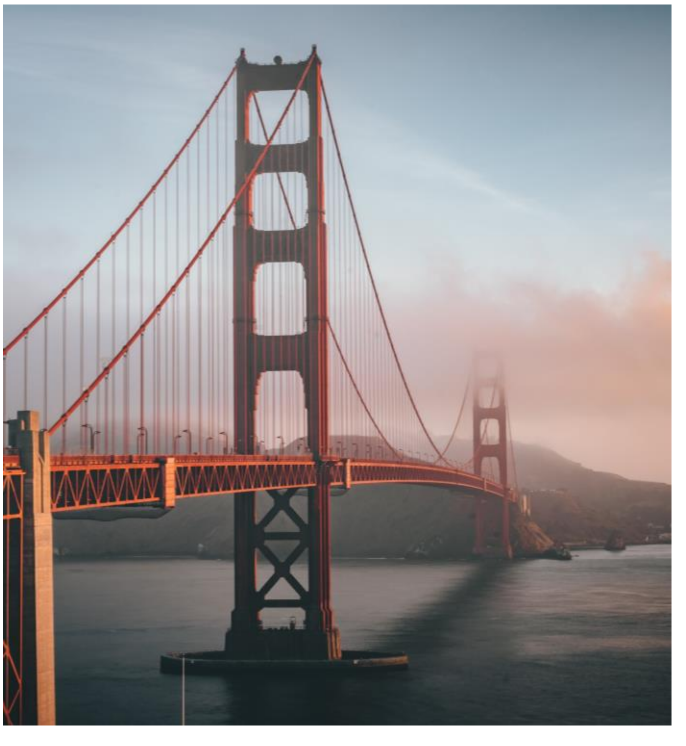
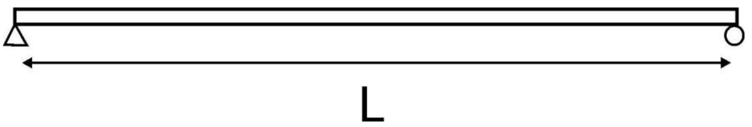
SINGLE MODE APPROXIMATION



Approach for continuous systems



$w(x) = \xi(t).f(x)$ Displacement of the beam (first mode shape approximation)

$\xi(t)$ displacement at a reference point $x = \xi$
$f(x)$ shape of the first mode shape of the beam with $f(x = \xi) = 1$

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Approach for continuous systems

$$w(x) = \xi(t).f(x)$$

$$E_s = \frac{1}{2}\int_0^L EIw(x)''^2\,dx = \frac{1}{2}\xi(t)^2\int_0^L EIf''(x)^2\,dx = \frac{1}{2}\tilde{k}\xi(t)^2$$ Strain energy

$$\tilde{k} = \int_0^L EIf''(x)^2\,dx = \vartheta\frac{EI}{L^3}$$

$$E_k = \frac{1}{2}\int_0^L \rho\dot{w}(x)^2\,dx = \frac{1}{2}\dot{\xi}(t)^2\int_0^L \rho f(x)^2\,dx = \frac{1}{2}\tilde{m}\dot{\xi}(t)^2$$ Kinetic energy

$$\tilde{m} = \int_0^L \rho f(x)^2\,dx = \phi_M \rho L$$

$$W_d = \int_0^L p(x)\,w(x)\,dx = \xi\int_0^L p(x)\,f(x)\,dx = \tilde{p}\,\xi$$

$$\tilde{p} = \int_0^L pf(x)\,dx = \phi_L\,pL$$

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Approach for continuous systems

$$\boxed{\tilde{m}\ddot{\xi} + \tilde{k}\xi = \tilde{p}}$$

$$\tilde{k} = \vartheta \frac{EI}{L^3}$$

$$\tilde{m} = \phi_M \rho L$$

$$\tilde{p} = \phi_L\, pL$$

Loading and support conditions (Reference point at l/2)	Load factor ϕ_L	Mass factor ϕ_M: Lumped mass	Mass factor ϕ_M: Distributed mass	Effective beam stiffness k	Stiffness factor ϑ
P = pl	0.637	—	0.5	$\frac{384\,EI}{5 \cdot l^3}$	48.7
P	1.0	1.0	0.5	$\frac{48\,EI}{l^3}$	48.7
P = pl	0.595	—	0.479	$\frac{185\,EI}{l^3}$	113.9
P	1.0	1.0	0.479	$\frac{107\,EI}{l^3}$	113.9
P = pl	0.523	—	0.396	$\frac{384\,EI}{l^3}$	198.5
P	1.0	1.0	0.396	$\frac{192\,EI}{l^3}$	198.5

Figure A.9: Equivalent substitute SDOF parameters for single span beams with various support and load conditions (see explanation in the text)

Vibration problems in structures, H. Bachman, 1995

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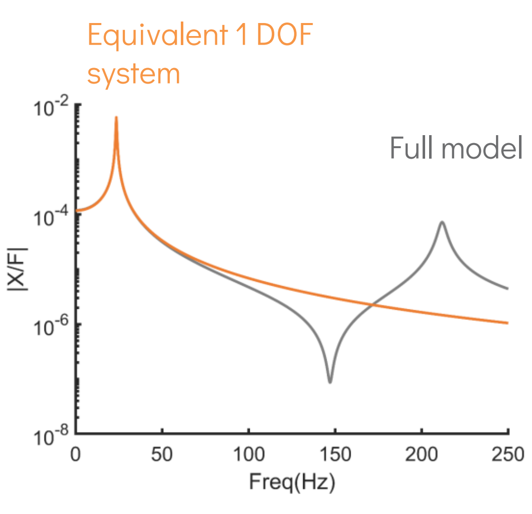
Example of a building



Collocated transfer function

			ϕ_M		ϑ
P	1.0	1.0	0.5	$\frac{48\,EI}{l^3}$	48.7

$$\tilde{k} = \vartheta \frac{EI}{L^3}$$

$$\tilde{m} = \phi_M \rho L$$

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Finite element approach

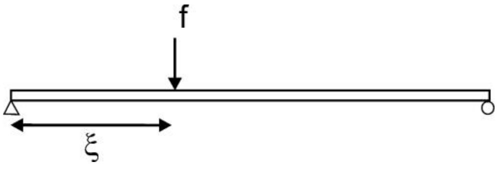
$$M\ddot{x} + C\dot{x} + Kx = f$$

$x = \psi_i z_i(t)$ Single mode approximation

$$\psi_i^T M \psi_i \ddot{z}_i + \psi_i^T C \psi_i \dot{z}_i + \psi_i^T K \psi_i z_i = \psi_i^T F$$

$$\mu_i \ddot{z}_i + 2\mu_i \xi_i \omega_i \dot{z}_i + \mu_i \omega_i^2 z_i = F_i$$

Particular case of a point load at reference position



$$x(\xi) = \psi_i(\xi) z_i(t)$$

$$F_i = \psi_i(\xi) f$$

$$\frac{\mu_i}{\psi_i^2(\xi)}\ddot{x}(\xi) + \frac{2\mu_i\omega_i\xi_i}{\psi_i^2(\xi)}\dot{x}(\xi) + \frac{\mu_i\omega_i^2}{\psi_i^2(\xi)}x(\xi) = f$$

Equation of motion projected in the modal domain

Position dependent scaling

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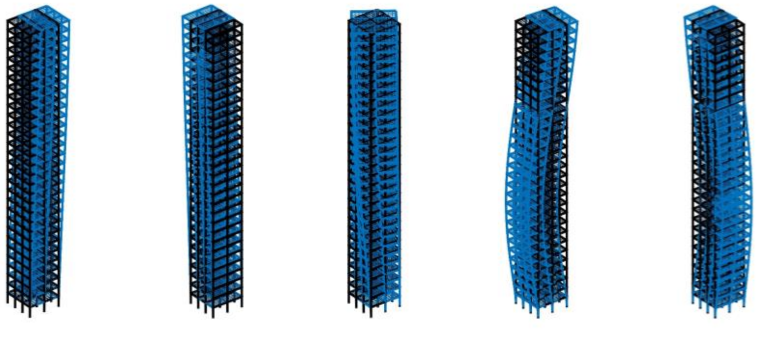
Building mode shapes

Mode 1 (0.24 Hz) Mode 2 (0.27 Hz) Mode 3 (0.40 Hz) Mode 4 (0.79 Hz) Mode 5 (0.89 Hz)



Mode shapes are generally ‘mass normalized’ : $\mu_i = 1$

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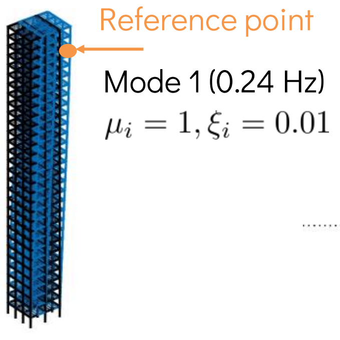
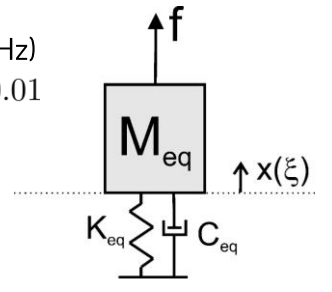
Exemple of a building

$$M_{eq}\ddot{x}(\xi) + C_{eq}\dot{x}(\xi) + K_{eq}x(\xi) = f$$

$$M_{eq} = \frac{\mu_i}{\psi_i^2(\xi)} \qquad C_{eq} = \frac{2\mu_i\omega_i\xi_i}{\psi_i^2(\xi)} \qquad K_{eq} = \frac{\mu_i\omega_i^2}{\psi_i^2(\xi)}$$





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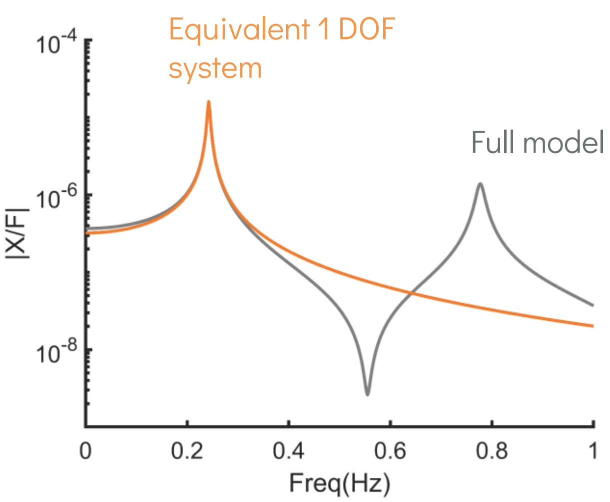
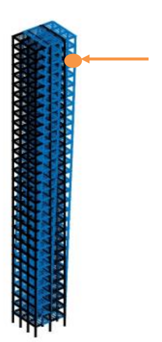
Exemple of a building



Collocated transfer function



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