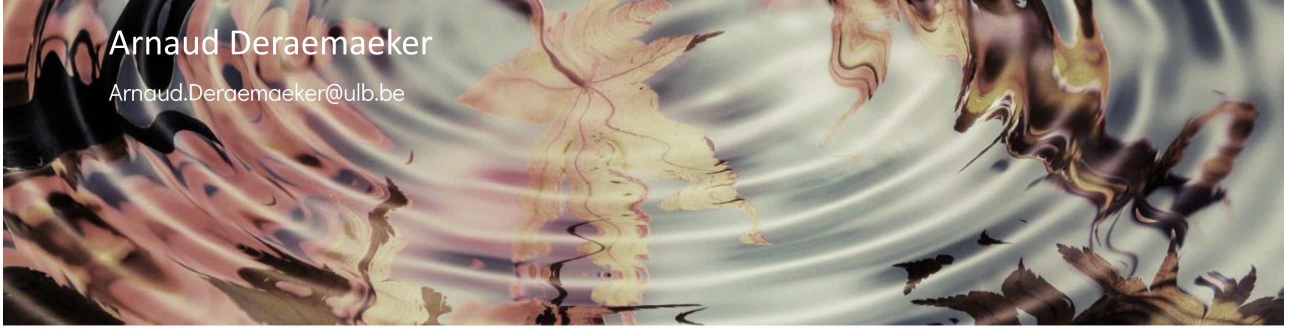
MDOF SYSTEMS



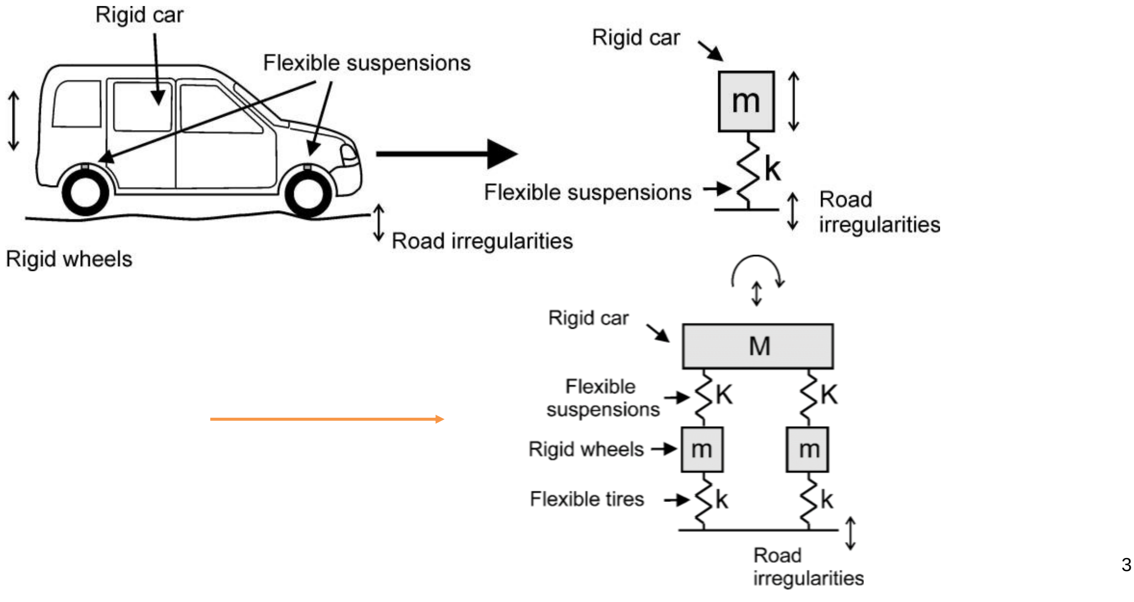
Arnaud Deraemaeker
Arnaud.Deraemaeker@ulb.be

1

MDOF SYSTEMS IN REAL LIFE

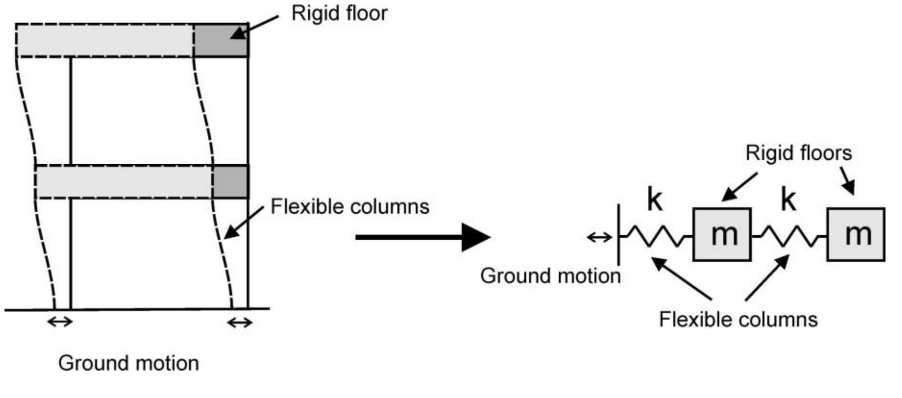
2

From SDOF to MDOF



3

MDOF systems in real life

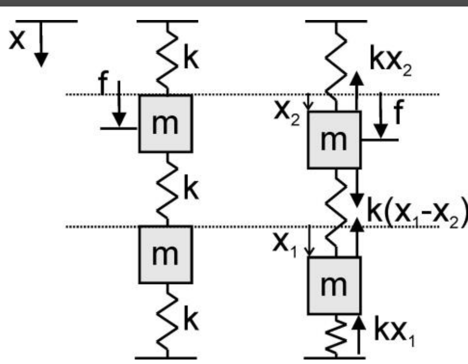


4

UNDAMPED RESPONSE

5

Equations of motion



$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - k\left(x_1 - x_2\right) \\ m\ddot{x}_2 &= k\left(x_1 - x_2\right) - kx_2 + f \end{aligned}$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

$$\boxed{M\ddot{x} + Kx = F}$$

Mass matrix

Stiffness matrix

6

6

Free response

$$\boxed{M\ddot{x} + Kx = 0}$$

$$\left\{ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right\} = \left\{ \begin{array}{c} A_1 \\ A_2 \end{array} \right\} e^{rt} = \psi e^{rt}$$

$$\left(K + r^2 M\right)\psi = 0$$

Admits a non trivial solution if

$$det(K + r^2 M) = 0$$

r^2 is negative (K and M are positive definite matrices)

$$r^2 = -\omega^2$$

$$\longrightarrow \boxed{\left(K - \omega^2 M\right)\psi = 0}$$

7

Free response

$$\left(K - \omega^2 M\right)\psi = 0$$

Generalized eigenvalue problem ($-\omega^2$)

$$det(K - \omega^2 M) = 0$$

If the system has *n* degrees of freedom, there exist *n* values of $-\omega^2$ for which this equation is satisfied. These are the *n* eigenvalues which correspond to *n* eigenfrequencies

n eigen vectors ψ are associated to these eigenfrequencies. They correspond to the *n* mode shapes of the structure

The general solution is written in the form:

$$x(t) = \sum_{i=1}^{n} \left(Z_{i1} cos(\omega_i t) + Z_{i2} sin(\omega_i t)\right) \psi_i$$

8

Mode shapes orthogonality

Property :

$$\boxed{\begin{aligned}\psi_i^T M \psi_j &= \delta_{ij}\mu_i \\ \psi_i^T K \psi_j &= \delta_{ij}\mu_i\omega_i^2\end{aligned}}$$

Proof :

$$\left(K - \omega_i^2 M\right)\psi_i = 0 \qquad (1)$$
$$\left(K - \omega_j^2 M\right)\psi_j = 0 \qquad (2)$$
$$\omega_i \neq \omega_j$$

Premultiply (1) by ψ_j^T ,(2) by ψ_i^T and substract taking into account symmetry of K ($\psi_i^T K \psi_j = \psi_j^T K \psi_i$) and M ($\psi_i^T M \psi_j = \psi_j^T M \psi_i$)

$$\longrightarrow \quad \psi_j^T M \psi_i (\omega_j^2 - \omega_i^2) = 0 \qquad i \neq j$$

$$\longrightarrow \quad \psi_j^T M \psi_i = 0 \qquad i \neq j$$

Mode shapes orthogonality

$$\psi_j^T M \psi_i = 0 \qquad i \neq j$$

Define $\mu_i = \psi_i^T M \psi_i \longrightarrow \psi_i^T M \psi_j = \delta_{ij}\mu_i$

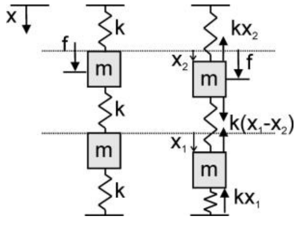
$$K\psi_i = \omega_i^2 M \psi_i \longrightarrow \psi_i^T K \psi_j = \delta_{ij}\mu_i\omega_i^2$$

Matrix notation

$$\Psi = \left[\begin{array}{cccc}\psi_1 & \psi_2 & \ldots & \psi_n\end{array}\right]$$

$$\longrightarrow \quad \boxed{\begin{aligned}\Psi^T M \Psi &= diag(\mu_i) \\ \Psi^T K \Psi &= diag(\mu_i\omega_i^2)\end{aligned}}$$

Example of a 2 DOFs system



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

$$\boxed{\left(K - \omega^2 M\right) \psi = 0}$$

$$det(K - \omega^2 M) = det \begin{pmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{pmatrix} = 0$$

$$(2k - \omega^2 m)(2k - \omega^2 m) - k^2 = m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$$

Second order equation in ω^2 →
$$\begin{aligned} \omega_1^2 &= k/m \\ \omega_2^2 &= 3k/m \end{aligned}$$

Example of a 2 DOFs system

$$\boxed{\left(K - \omega^2 M\right) \psi = 0}$$

$$(K - \omega^2 M) = \begin{bmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{bmatrix}$$

$$\psi = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$$

For $\omega_1^2 = k/m$

$$\left(2k - \frac{k}{m} m\right) A_1 - kA_2 = 0$$
$$kA_1 = kA_2 \Rightarrow A_1 = A_2$$

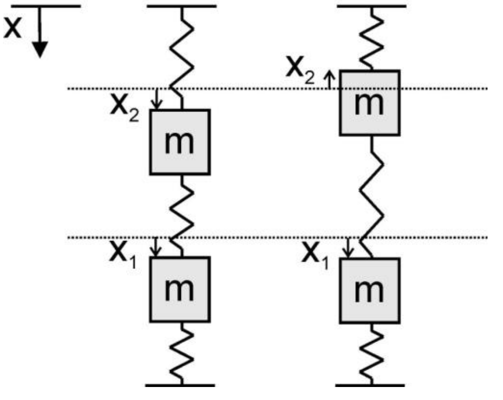
For $\omega_2^2 = 3k/m$

$$\left(2k - \frac{3k}{m} m\right) A_1 - kA_2 = 0$$
$$-kA_1 = kA_2 \Rightarrow A_1 = -A_2$$

Example of a 2 DOFs system

$$\omega_1^2 = k/m \qquad \psi_1 = \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}$$

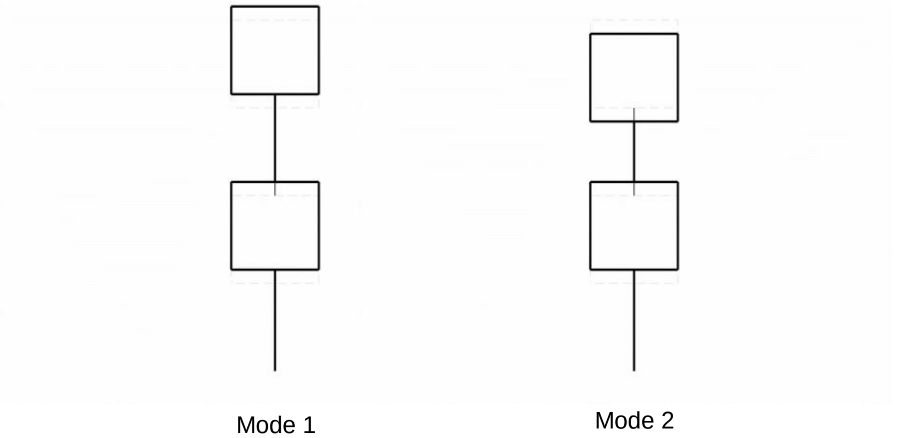
$$\omega_2^2 = 3k/m \qquad \psi_2 = \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\}$$



13

13

Example of a 2 DOFs system



Mode 1 Mode 2

14

14

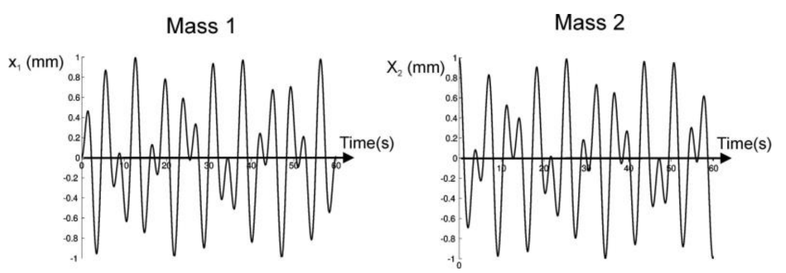
Example of a 2 DOFs system

$$\left\{ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right\} = (Z_{11}\cos\omega_1 t + Z_{12}\sin\omega_1 t)\left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} + (Z_{21}\cos\omega_2 t + Z_{22}\sin\omega_2 t)\left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\}$$

Assume the following initial conditions

$$\left\{ \begin{array}{c} x_1(0) \\ x_2(0) \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 1mm \end{array} \right\} \qquad \dot{x_1}(0) = \dot{x_2}(0) = 0$$

$$\longrightarrow \left\{ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right\} = \left(\begin{array}{c} \frac{1}{2}\cos\omega_1 t - \frac{1}{2}\cos\omega_2 t \\ \frac{1}{2}cos\omega_1 t + \frac{1}{2}cos\omega_2 t \end{array} \right) (mm)$$



15

15

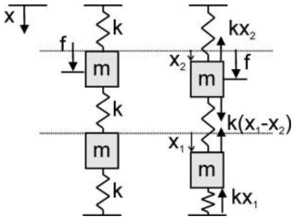
Resonance of MDOF systems

https://youtu.be/OaXSmPgl1os

16

16

Harmonic excitation



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

$$M\ddot{x} + Kx = f \qquad \begin{aligned} x(t) &= Xe^{i\omega t} \\ f(t) &= Fe^{i\omega t} \end{aligned}$$

$$\downarrow$$

$$\boxed{\left(K - \omega^2 M\right) X = F}$$

17

Example of a 2 DOFs system

Response to harmonic excitation $\begin{aligned} x_1(t) &= X_1 e^{i\omega t} \\ x_2(t) &= X_2 e^{i\omega t} \end{aligned}$ $\quad f(t) = Fe^{i\omega t}$

$$\begin{bmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

$$X_1/F = \frac{k}{\boxed{(2k - \omega^2 m)^2 - k^2}} \longrightarrow \begin{aligned} \omega_1^2 &= k/m \\ \omega_2^2 &= 3k/m \end{aligned}$$

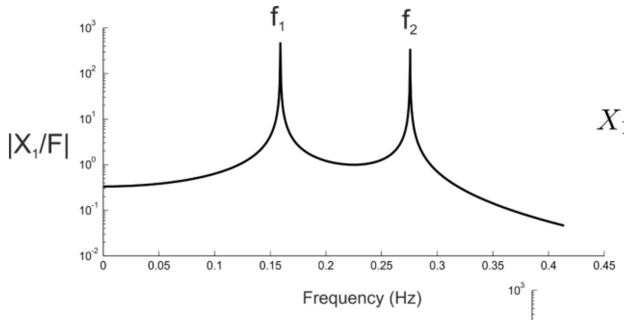
Resonance

$$X_2/F = \frac{\boxed{2k - \omega^2 m}}{(2k - \omega^2 m)^2 - k^2} \longrightarrow \omega_0^2 = 2k/m$$

Anti-resonance

18

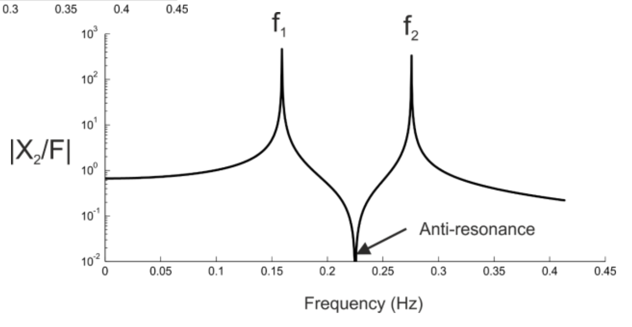
Example of a 2 DOFs system



$$X_1/F = \frac{-k}{(2k-\omega^2 m)^2 - k^2}$$

(m=1kg, k = 1N/m)

$$X_2/F = \frac{2k-\omega^2 m}{(2k-\omega^2 m)^2 - k^2}$$



19

Projection in the modal basis

$$\boxed{M\ddot{x} + Kx = F}$$

Projection in the modal basis

$$x(t) = \sum_{i=1}^{n} z_i(t)\psi_i \longrightarrow x = \Psi z$$

$$M\Psi\ddot{z} + K\Psi z = F$$
$$\Psi^T M\Psi\ddot{z} + \Psi^T K\Psi z = \Psi^T F$$
$$diag(\mu_i)\ddot{z} + diag(\mu_i\omega_i^2)z = \Psi^T F$$

n independent equations of the type

$$\longrightarrow \boxed{\mu_i\ddot{z}_i + \mu_i\omega_i^2 z_i = F_i}$$

20

Projection in the modal basis

$$x(t) = \sum_{i=1}^{n} z_i(t)\psi_i$$

The solution can be obtained by solving a set of n independent equations of the type

$$\boxed{\mu_i \ddot{z}_i + \mu_i \omega_i^2 z_i = F_i}$$

This equation corresponds to the equation of motion of a sdof system with

- a mass μ_i, called the modal mass
- a stiffness $\mu_i\omega_i^2$
- a natural frequency $\omega_i = 2\pi f_i$
- a force $F_i = \psi_i^T F$ (modal excitation)

21

Harmonic excitation : modal basis solution

$$z_i(t) = Z_i e^{i\omega t} \qquad X(\omega) = \sum_{i=1}^{n} Z_i(\omega)\psi_i \quad \text{or} \quad X = \Psi Z$$

$$\left(\Psi^T K \Psi - \omega^2 \Psi^T M \Psi\right) Z = \Psi^T F$$

$$\left(\begin{bmatrix} \mu_1\omega_1^2 & 0 & \dots & 0 \\ 0 & \mu_2\omega_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mu_n\omega_n^2 \end{bmatrix} - \omega^2 \begin{bmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mu_n \end{bmatrix} \right) \begin{Bmatrix} Z_1 \\ Z_2 \\ \dots \\ Z_n \end{Bmatrix} = \begin{Bmatrix} \psi_1^T F \\ \psi_2^T F \\ \dots \\ \psi_n^T F \end{Bmatrix}$$

$$\boxed{Z_i(\omega) = \psi_i^T F \frac{1}{\mu_i(\omega_i^2 - \omega^2)}}$$

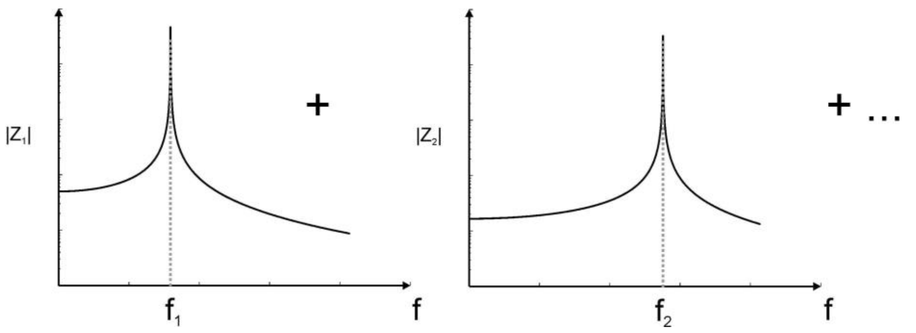
sdof oscillator solution

22

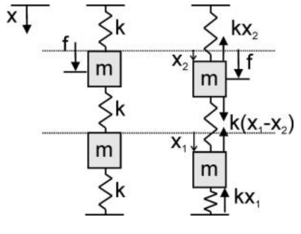
Harmonic excitation : modal basis solution

The solution is the sum of sdof oscillators :

$$X(\omega) = \sum_{i=1}^{n} Z_i(\omega)\psi_i = \sum_{i=1}^{n} \frac{\psi_i^T F \psi_i}{\mu_i(\omega_i^2 - \omega^2)}$$



Example of a 2 DOFs system



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

$$\Psi = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

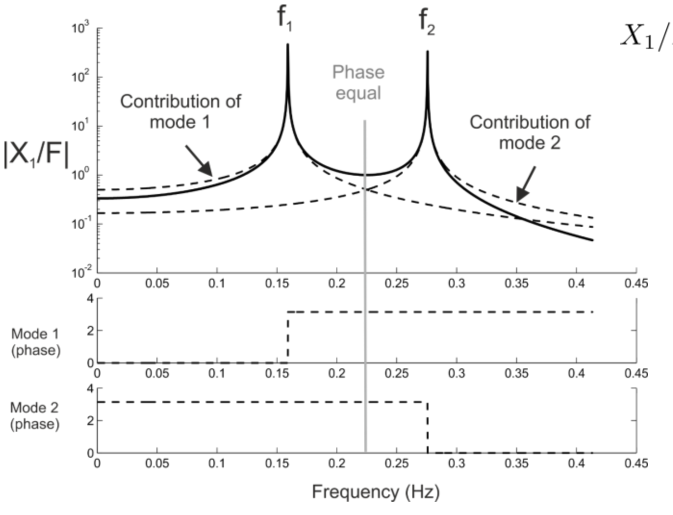
$$\Psi^T M \Psi = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix} \Rightarrow \mu_1 = \mu_2 = 2m$$

$$\Psi^T F = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{Bmatrix} 0 \\ F \end{Bmatrix} = \begin{Bmatrix} F \\ -F \end{Bmatrix}$$

$$F = 1 \rightarrow \quad Z_1 = \frac{1}{2m(\frac{k}{m} - \omega^2)} \qquad Z_2 = \frac{-1}{2m(\frac{3k}{m} - \omega^2)}$$

Example of a 2 DOFs system

$$X(\omega) = \left\{ \begin{array}{c} X_1(\omega) \\ X_2(\omega) \end{array} \right\} = \frac{1}{2m(\frac{k}{m} - \omega^2)} \left\{ \begin{array}{c} \boxed{1} \\ 1 \end{array} \right\} + \frac{-1}{2m(\frac{3k}{m} - \omega^2)} \left\{ \begin{array}{c} \boxed{1} \\ -1 \end{array} \right\}$$



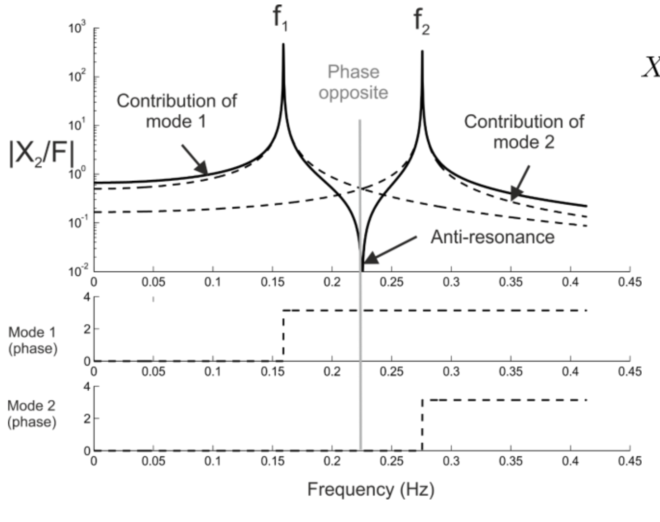
$$X_1/F = \frac{-k}{(2k - \omega^2 m)^2 - k^2}$$

(m=1kg, k = 1N/m)

25

Example of a 2 DOFs system

$$X(\omega) = \left\{ \begin{array}{c} X_1(\omega) \\ X_2(\omega) \end{array} \right\} = \frac{1}{2m(\frac{k}{m} - \omega^2)} \left\{ \begin{array}{c} 1 \\ \boxed{1} \end{array} \right\} + \frac{-1}{2m(\frac{3k}{m} - \omega^2)} \left\{ \begin{array}{c} 1 \\ \boxed{-1} \end{array} \right\}$$



$$X_2/F = \frac{2k - \omega^2 m}{(2k - \omega^2 m)^2 - k^2}$$

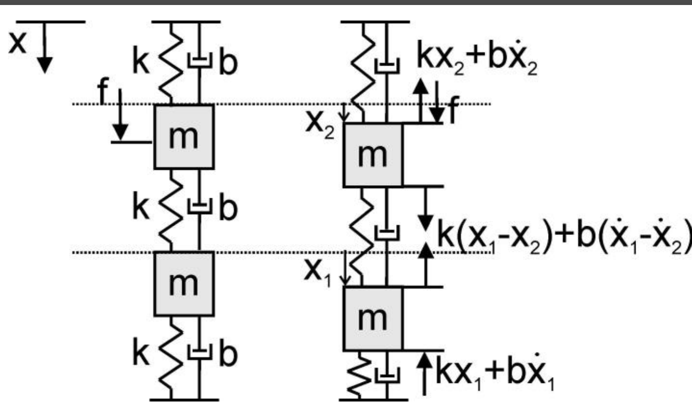
(m=1kg, k = 1N/m)

26

DAMPED RESPONSE

27

Equations of motion



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2b & -b \\ -b & 2b \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

$$M\ddot{x} + C\dot{x} + Kx = f$$

Damping matrix

28

28

Free response

$$\left\{ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right\} = \left\{ \begin{array}{c} A_1 \\ A_2 \end{array} \right\} e^{rt} = \psi e^{rt}$$

$$\left(K + rC + r^2 M\right) \psi = 0$$

Non trivial solution if

$$det(K + rC + r^2 M) = 0$$

•Complex roots of the characteristic equation
-> Oscillatory functions with exponential envelope

•Complex eigen vectors = complex modeshapes
-> Not often used in practice in vibrations

29

29

Projection in the basis of the real mode shapes

Mode shapes of the conservative system :

$$\boxed{\left(K - \omega^2 M\right) \psi = 0}$$

Projection on the real modal basis:

$$M\ddot{x} + C\dot{x} + Kx = F \qquad x(t) = \sum_{i=1}^{n} z_i(t)\psi_i \quad \text{or} \quad x = \Psi z$$

$$\Psi^T M \Psi \ddot{z} + \Psi^T C \Psi \dot{z} + \Psi^T K \Psi z = \Psi^T F$$

In general $\Psi^T C \Psi$ is not diagonal and the equations remain coupled but ...

30

30

Projection in the basis of the real mode shapes

Rayleigh damping:

$$C = \alpha K + \beta M$$

$$\longrightarrow \quad \Psi^T C \Psi = \Psi^T(\alpha K + \beta M)\Psi = diag(\alpha\mu_i\omega_i^2 + \beta\mu_i)$$

Often used as a simplifying assumption to decouple the equations but does not have a physical meaning

Modal damping

When damping is small, off-diagonal terms can be neglected leading to:

$$\Psi^T C \Psi = \begin{bmatrix} 2\mu_1\xi_1\omega_1 & 0 & \dots & 0 \\ 0 & 2\mu_2\xi_2\omega_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 2\mu_n\xi_n\omega_n \end{bmatrix}$$

ξ_i is the modal damping of mode i

31

31

Projection in the basis of the real mode shapes

$$diag(\mu_i)\ddot{z} + diag(2\xi_i\mu_i\omega_i)\dot{z} + diag(\mu_i\omega_i^2)z = \Psi^T F$$

n independent equations of the type

$$\longrightarrow \quad \boxed{\mu_i\ddot{z}_i + 2\mu_i\xi_i\omega_i\dot{z}_i + \mu_i\omega_i^2 z_i = F_i}$$

This equation corresponds to the equation of motion of a sdof system with

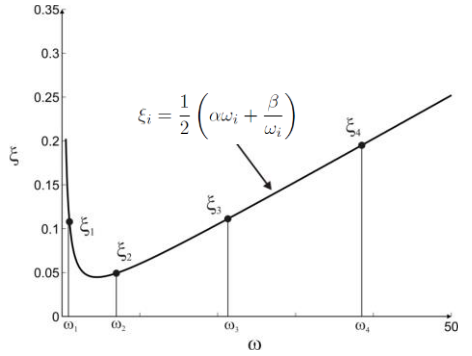
- a mass μ_i, called the modal mass
- a stiffness $\mu_i\omega_i^2$
- a damping coefficient ξ_i
- a natural frequency $\omega_i = 2\pi f_i$
- a force $F_i = \psi_i^T F$ (modal excitation)

32

32

Link between Rayleigh and modal damping

$$\underbrace{diag(\alpha\mu_i\omega_i^2 + \beta\mu_i)}_{\text{Rayleigh damping}} = \underbrace{diag(2\xi_i\mu_i\omega_i)}_{\text{Modal damping}} \quad \Rightarrow \xi_i = \frac{1}{2}\left(\alpha\omega_i + \frac{\beta}{\omega_i}\right)$$

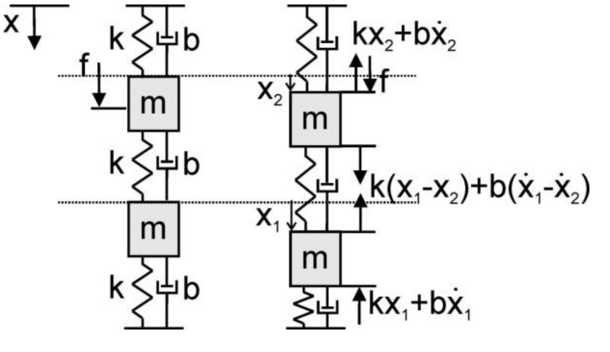


- Only two parameters to define the damping of all modes
  - ->Overestimation at low and high frequencies
  - ->Represents accurately the damping of two modes only

33

33

Harmonic excitation



$$M\ddot{x} + C\dot{x} + Kx = f \qquad x(t) = Xe^{i\omega t} \qquad f(t) = Fe^{i\omega t}$$

$$\boxed{\left(K + i\omega C - \omega^2 M\right) X = F}$$

34

34

Example of a 2 DOFs system

Response to harmonic excitation $x_1(t) = X_1 e^{i\omega t}$ $f(t) = Fe^{i\omega t}$
$x_2(t) = X_2 e^{i\omega t}$

$$\begin{bmatrix} 2k + 2i\omega b - \omega^2 m & -(k + i\omega b) \\ -(k + i\omega b) & 2k + 2i\omega b - \omega^2 m \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

$$X_1/F = \frac{k + i\omega b}{\boxed{(2k + 2i\omega b - \omega^2 m)^2 - (k + i\omega b)^2}}$$

Damped resonances

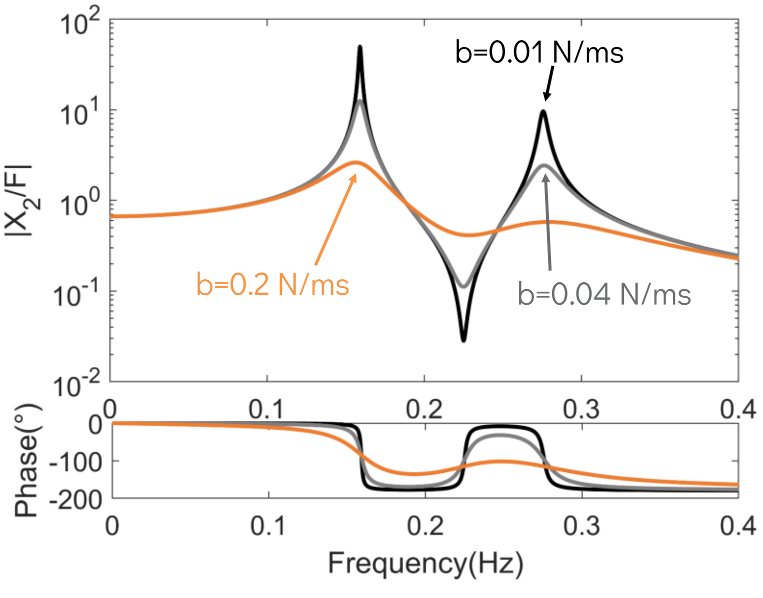
$$X_2/F = \frac{\boxed{2k + 2i\omega b - \omega^2 m}}{(2k + 2i\omega b - \omega^2 m)^2 - (k + i\omega b)^2}$$

No strict anti-resonance

35

35

Example of a 2 DOFs system



(m=1kg, k = 1N/m)

36

36

Harmonic excitation : modal basis solution

$$\left(K + i\omega C - \omega^2 M\right) X = F$$

Projection on modal basis $X(\omega) = \sum_{j=1}^{n} Z_j(\omega)\psi_j \quad X = \Psi Z$

$$\left(\Psi^T K \Psi + i\omega \Psi^T C \Psi - \omega^2 \Psi^T M \Psi\right) Z = \Psi^T F$$

Modal damping hypothesis (small damping)

$$\mu_j(\omega_j^2 - \omega^2 + 2i\xi_j\omega\omega_j)Z_j = F_j$$

$$Z_j(\omega) = \frac{\psi_j^T F}{\mu_j(\omega_j^2 - \omega^2 + 2i\xi_j\omega\omega_j)}$$

n decoupled equations

$$\boxed{X(\omega) = \sum_{j=1}^{n} \frac{\psi_j^T F \psi_j}{\mu_j(\omega_j^2 - \omega^2 + 2i\xi_j\omega\omega_j)}}$$

Sum of damped sdof oscillators

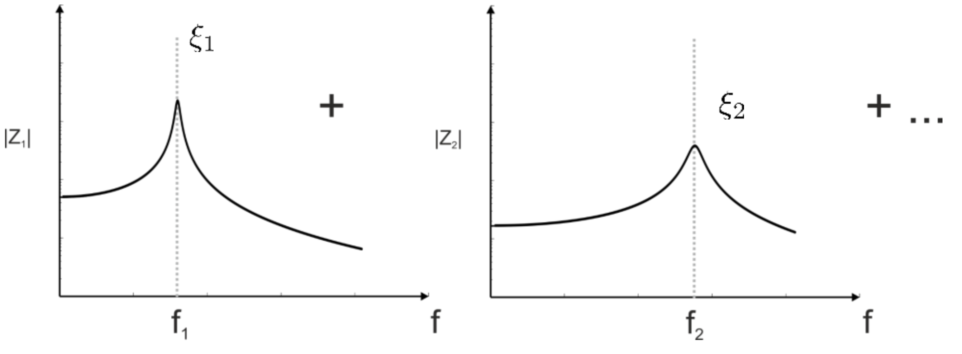
37

37

Harmonic excitation : modal basis solution

The solution is the sum of damped sdof oscillators :

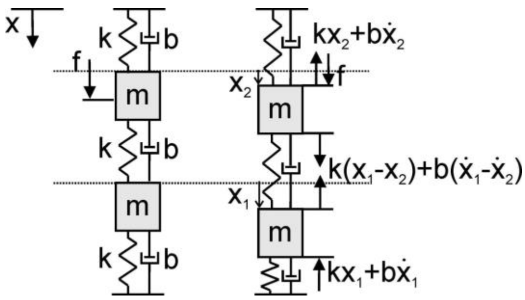
$$X(\omega) = \sum_{j=1}^{n} \frac{\psi_j^T F \psi_j}{\mu_j(\omega_j^2 - \omega^2 + 2i\xi_j\omega\omega_j)}$$



38

38

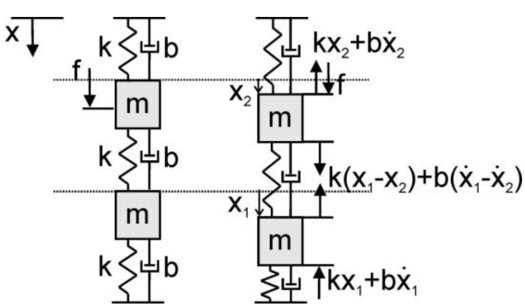
Example of a 2 DOFs system



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2b & -b \\ -b & 2b \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

$$M\ddot{x} + C\dot{x} + Kx = f$$

Example of a 2 DOFs system : projection in the modal basis



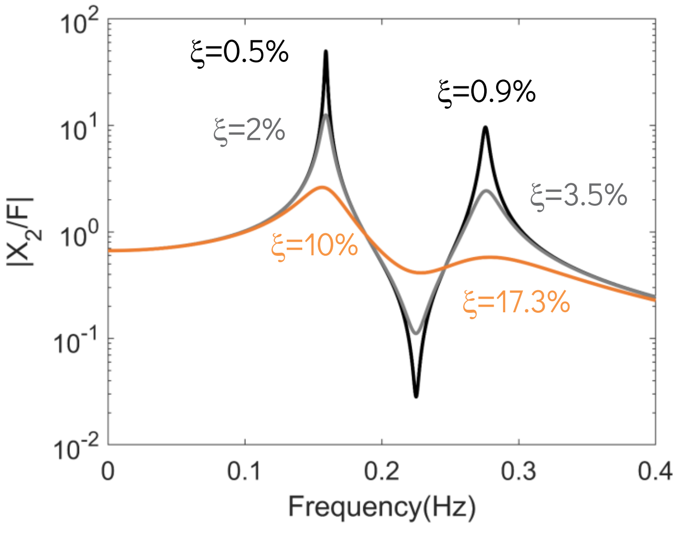
$$\Psi^T C \Psi = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2b & -b \\ -b & 2b \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2b & 0 \\ 0 & 6b \end{bmatrix}$$

$$\xi_1 = \frac{b}{2\sqrt{km}}$$

$$\xi_2 = \frac{\sqrt{3}b}{2\sqrt{km}} = \sqrt{3}\,\xi_1$$

Modal damping coefficients

Example of a 2 DOFs system



k=1 N/m, m=1kg,

b=0.01 N/ms

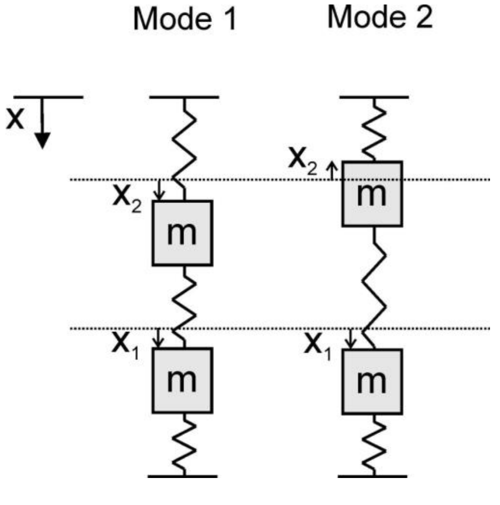
b=0.04 N/ms

b=0.2 N/ms

41

41

Example of a 2 DOFs system

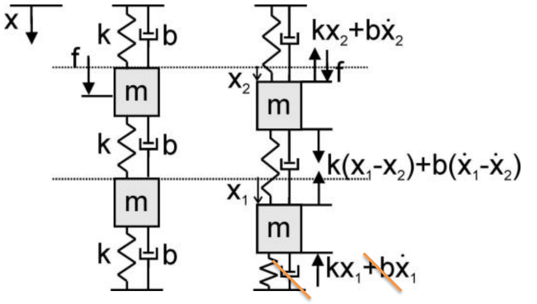


The second mode is more damped because in the first mode, the dashpot between masses 1 and 2 is not dissipating energy

42

42

Validity of modal damping hypothesis

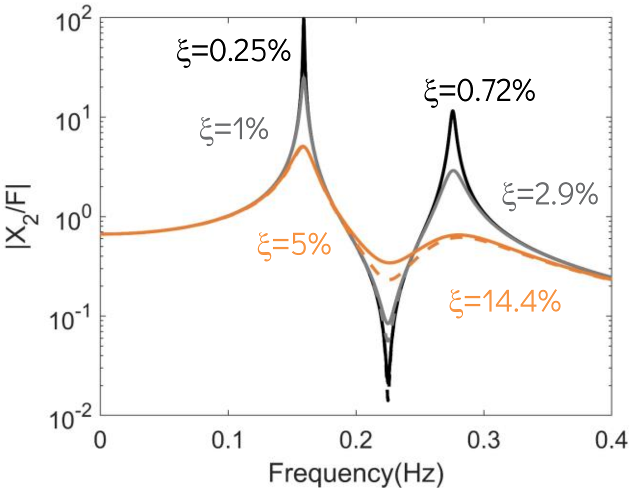


$$\Psi^T C \Psi = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b & -b \\ -b & 2b \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} b & -b \\ -b & 5b \end{bmatrix}$$

Neglect off-diagonal terms $\Psi^T C \Psi = \begin{bmatrix} b & 0 \\ 0 & 5b \end{bmatrix}$

$$\xi_1 = \frac{b}{4\sqrt{km}} \qquad \xi_2 = \frac{5}{\sqrt{3}}\xi_1$$

Validity of modal damping hypothesis



k=1 N/m, m=1kg,

b=0.01 N/ms

b=0.04 N/ms

b=0.2 N/ms

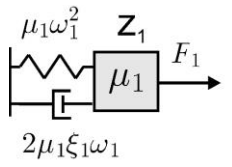
→ Modal damping hypothesis is valid when damping is small (ξ<10%), and away from anti-resonance

Comparison of exact (dotted line) and modal damping responses

Impulse response

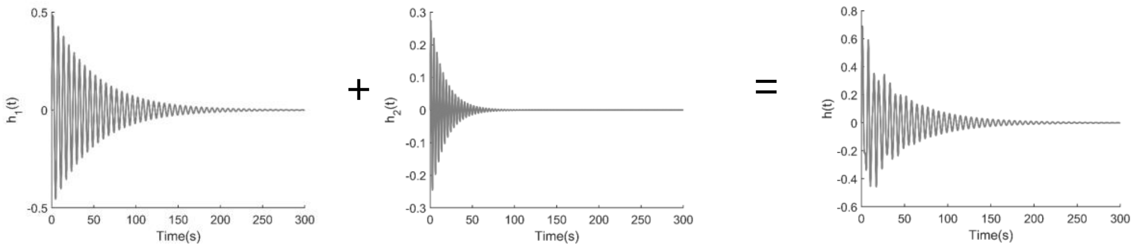
$$x(t) = \sum_{i=1}^{n} z_i(t)\psi_i$$

$$z_i(t) = F_i h_i(t) = F_i \frac{e^{-\xi_i \omega_i t}}{\mu_i \omega_{di}} \sin(\omega_{di} t)$$



...

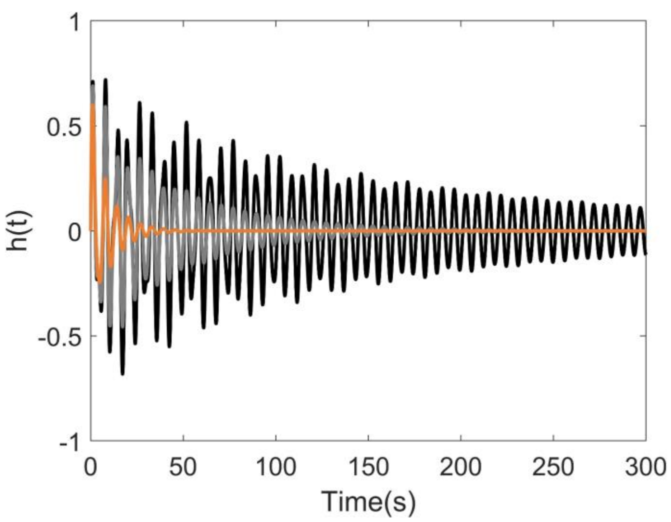
k=1 N/m, m=1kg,
b=0.04 N/ms



Higher frequency modes disappear faster in the impulse response 45

45

Impulse response



k=1 N/m, m=1kg,

b=0.01 N/ms

b=0.04 N/ms

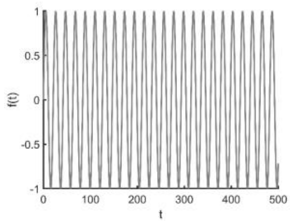
b=0.2 N/ms

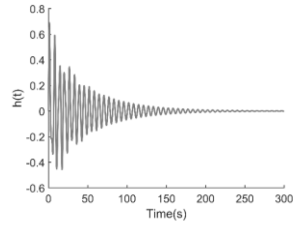
The impulse response decays faster with a higher damping. 46

46

Harmonic excitation below resonance

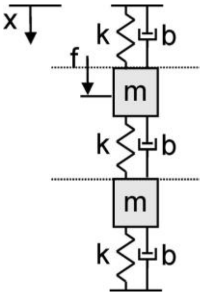
$f(t) = \sin(\omega t)$ $\quad * \quad$ $\mathrm{h(t)}$



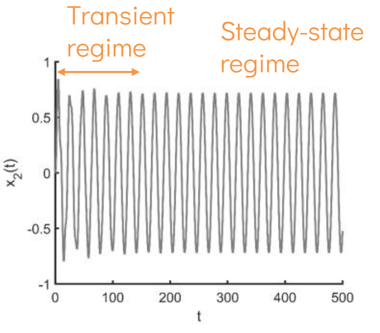


$$\omega = 0.3\ \omega_1$$

k=1 N/m, m=1kg,
b=0.04 N/ms



$=$

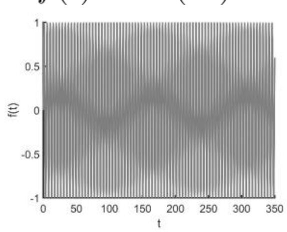


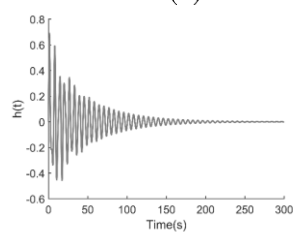
47

47

Harmonic excitation between the two resonance frequencies

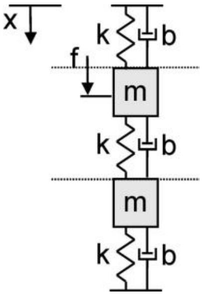
$f(t) = \sin(\omega t)$ $\quad * \quad$ $\mathrm{h(t)}$



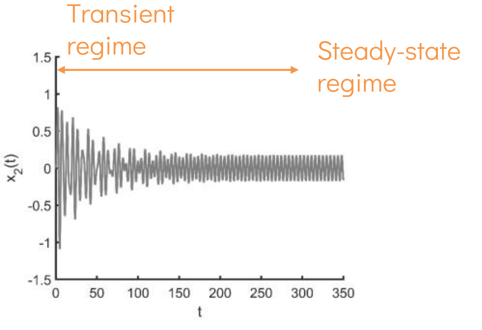


$$\omega = \frac{\omega_1 + \omega_2}{2}$$

k=1 N/m, m=1kg,
b=0.04 N/ms



$=$

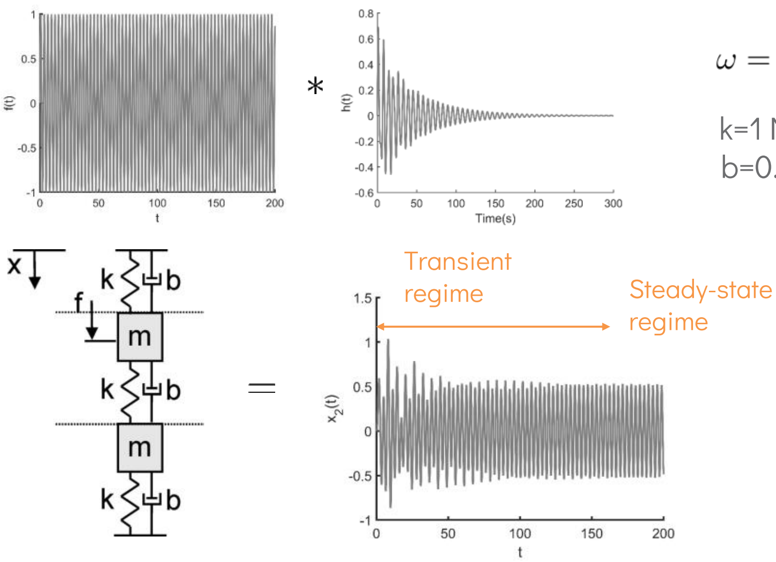


48

48

Harmonic excitation above the two resonance frequencies

$f(t) = \sin(\omega t)$ $\quad * \quad$ $\mathrm{h(t)}$



$\omega = 1.2\ \omega_2$

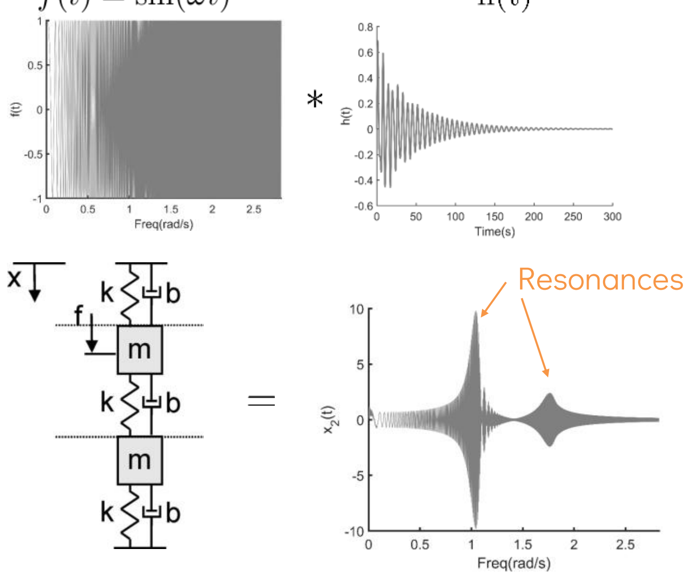
k=1 N/m, m=1kg,
b=0.04 N/ms

49

49

Sine sweep excitation

$f(t) = \sin(\omega t)$ $\quad * \quad$ $\mathrm{h(t)}$

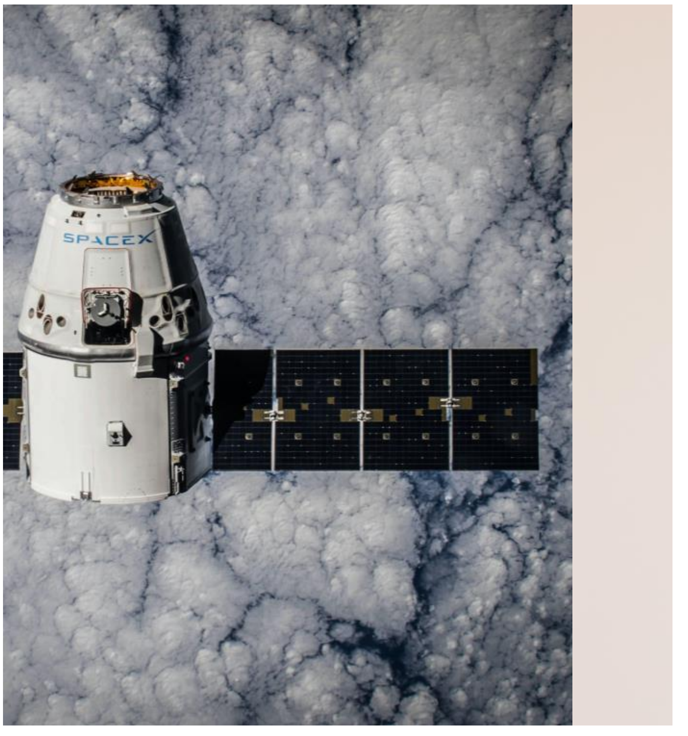


k=1 N/m, m=1kg,
b=0.04 N/ms

50

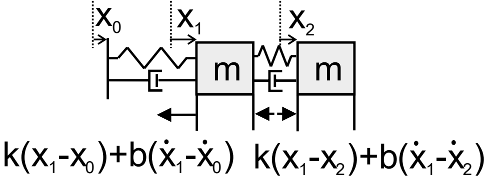
50

BASE EXCITATION



51

Base excitation of MDOF systems



Equations of motion:

$$\begin{aligned} m\ddot{x}_1 &= -k(x_1-x_0)-b(\dot{x}_1-\dot{x}_0)-k(x_1-x_2)-b(\dot{x}_1-\dot{x}_2) \\ m\ddot{x}_2 &= k(x_1-x_2)+b(\dot{x}_1-\dot{x}_2) \end{aligned}$$

$$\begin{aligned} x_{1r} &= x_1 - x_0 \\ x_{2r} &= x_2 - x_0 \end{aligned} \rightarrow \begin{aligned} m\ddot{x}_{1r}+2b\dot{x}_{1r}-b\dot{x}_{2r}+2kx_{1r}-kx_{2r} &= -m\ddot{x}_0 \\ m\ddot{x}_{2r}+b\dot{x}_{2r}-b\dot{x}_{1r}+kx_{2r}-kx_{1r} &= -m\ddot{x}_0 \end{aligned}$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_{1r} \\ \ddot{x}_{2r} \end{Bmatrix}+\begin{bmatrix} 2b & -b \\ -b & b \end{bmatrix}\begin{Bmatrix} \dot{x}_{1r} \\ \dot{x}_{2r} \end{Bmatrix}+\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_{1r} \\ x_{2r} \end{Bmatrix}=\begin{Bmatrix} -m\ddot{x}_0 \\ -m\ddot{x}_0 \end{Bmatrix}$$

52

52

Base excitation of MDOF systems

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_{1r} \\ \ddot{x}_{2r} \end{Bmatrix} + \begin{bmatrix} 2b & -b \\ -b & b \end{bmatrix} \begin{Bmatrix} \dot{x}_{1r} \\ \dot{x}_{2r} \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_{1r} \\ x_{2r} \end{Bmatrix} = \begin{Bmatrix} -m\ddot{x}_0 \\ -m\ddot{x}_0 \end{Bmatrix}$$

Matrix notations:

$$M\ddot{x}_r + C\dot{x}_r + Kx_r = -M\ddot{x}_b$$

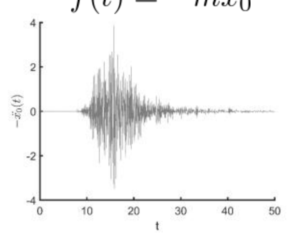
$$x_r = \begin{Bmatrix} x_1 - x_0 \\ x_2 - x_0 \end{Bmatrix} \qquad \ddot{x}_b = \begin{Bmatrix} \ddot{x}_0 \\ \ddot{x}_0 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ddot{x}_0 = T\,\ddot{x}_0$$

→ All developments for force excitation apply

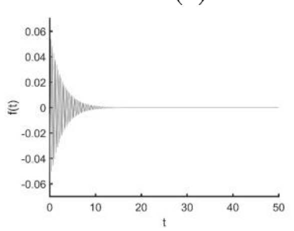
53

Santa Cruz earthquake (1990)

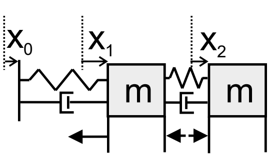
$f(t) = -m\ddot{x}_0$ h(t)



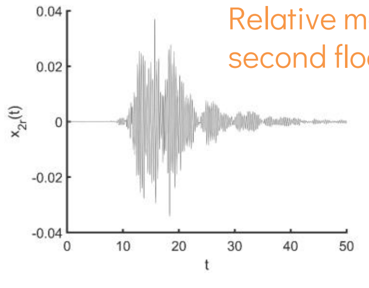
$*$



k=1 kN/m, m=1kg,
b=2 N/ms



$=$



Relative motion of second floor

$$f_1 = 3.11Hz$$
$$f_2 = 8.14Hz$$
$$\xi_1 = 1.95\%$$
$$\xi_2 = 5.12\%$$

54