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Session 2: MDOF systems

Exercise 1: Multiple DOFs System

Consider the following three degrees-of-freedom model:

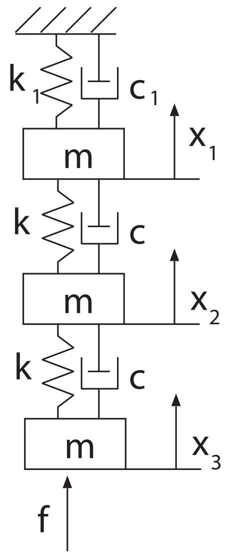


Figure 1: 3-DOFs system

A) Write the equations of motion in the time domain in a matrix form.

ANSWER:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c_1 + c & -c & 0 \\ -c & 2c & -c \\ 0 & -c & c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f \end{Bmatrix}$$

B) For the numerical values $m = 1\,kg$, $k = k_1 = 16\,N/m$, $c = c_1 = 0.1\,Ns/m$, compute the eigenfrequencies and the modeshapes of the conservative system. Represent the mode-shapes, and check the orthogonality conditions. .

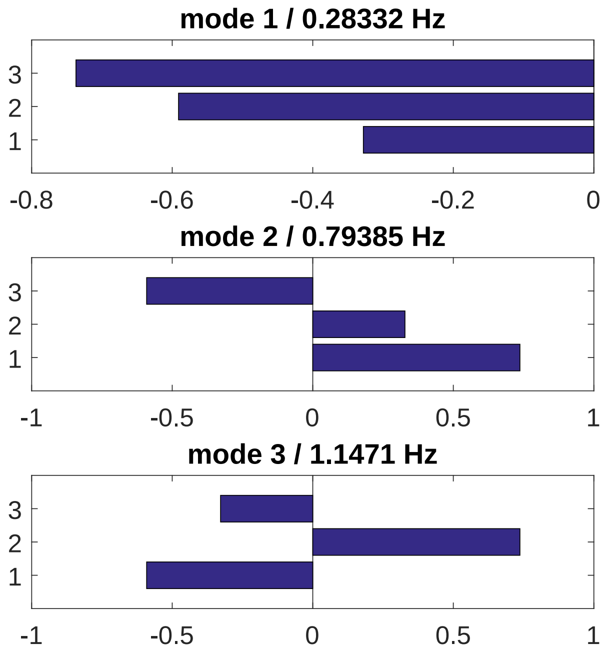
ANSWER:
Solve the problem:

$$\left(\mathbf{K} - \omega_i^2 \mathbf{M}\right) \psi_i = 0$$

The orthogonality conditions are given by

$$\psi_i \mathbf{K} \psi_i^T = \omega_i^2 \quad \psi_i \mathbf{M} \psi_i^T = \omega_i^2 \mu_i$$

```
clear all; close all;clc;

% Build matrices of the system
m=1; k=16; k1=16; c=0.1; c1=0.1;
K=[k1+k -k 0 ; -k 2*k -k ; 0 -k k];
M=[m 0 0 ; 0 m 0 ; 0 0 m];
C=[c1+c -c 0 ; -c 2*c -c ; 0 -c c];

% compute and represent modeshapes
[V,D]=eig(K,M);
f=1/(2*pi)*sqrt(diag(D)); % frequencies in Hz
figure;
subplot(3,1,1); barh(V(:,1)); title(['mode 1 / ' num2str(f(1)) ' Hz'])
subplot(3,1,2); barh(V(:,2)); title(['mode 2 / ' num2str(f(2)) ' Hz'])
subplot(3,1,3); barh(V(:,3)); title(['mode 3 / ' num2str(f(3)) ' Hz'])

% Check orthogonality
MU=diag(V'*M*V) % mass normalized
W2MU=diag(V'*K*V)
disp(MU)
disp(W2MU)
disp([W2MU./MU diag(D)]);
```



C) For the value of $k_1 = k$, compute the impulse response for x_3 by projecting the equations of motion in the modal basis. Represent the Bode diagram for the same coordinate x_3, and for the acceleration $\ddot{x}_3$. Is the modal damping hypothesis valid . Multiply by a factor 5 the damping coefficient of mode 2 and plot the Bode diagram for x_3 on the same graph as with the initial value of the damping. Comment.

ANSWER:

The projection in the modal basis results in three independent equations of the form

$$\mu_i(\ddot{z}_i + 2\xi_i\omega_i\dot{z}_i + \omega_i^2 z_i) = F_i$$

The force F_i exciting each mode is given by (the force f is only acting on x_3):

$$F = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}^T \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} V_{31} \\ V_{32} \\ V_{33} \end{Bmatrix}$$

Therefore we see that:

$$F_i = V_{3i}$$

For one mode, the impulse response corresponds to the impulse response of a 1DOF system

$$h_i(t) = \frac{e^{-\xi_i\omega_i t}}{\mu_i\omega_{di}} \sin(\omega_{di} t)$$

with

$$\omega_{di} = \omega_i\sqrt{1 - \xi_i^2}$$

The impulse response at point x_3 is therefore given by (one must multiply the impulse response both by F_i and the amplitude of the respective mode at x_3)

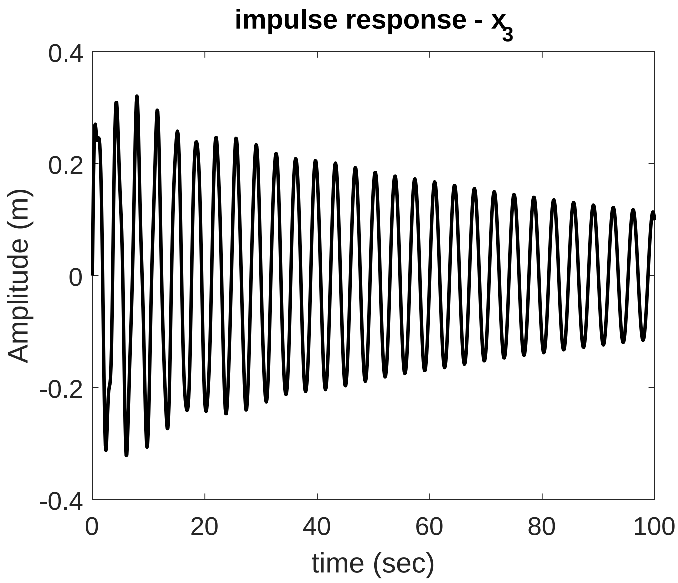
$$h(t) = \sum_{i=1}^{3} V_{3i}F_i h_i(t) = \sum_{i=1}^{3} V_{3i}^2 \frac{e^{-\xi_i\omega_i t}}{\mu_i\omega_{di}} \sin(\omega_{di} t)$$

Similarly, the response in the frequency domain is given by:

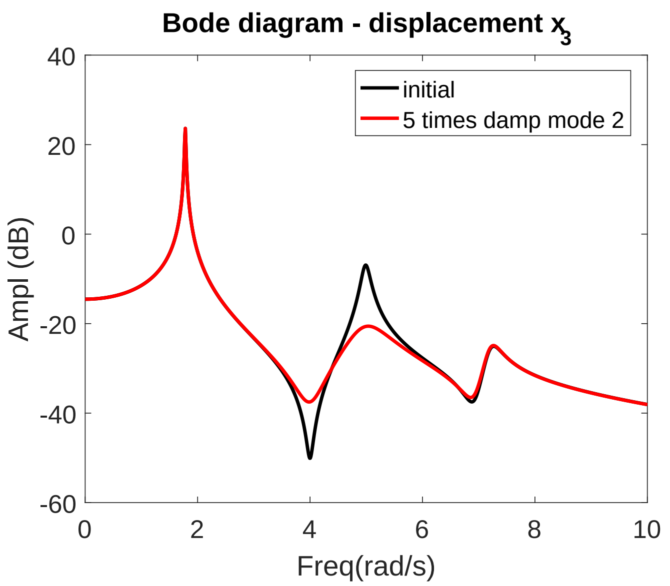
$$H_{33}(\omega) = \sum_{i=1}^{3} \frac{V_{3i}^2}{\omega_i^2 - \omega^2 + 2j\xi_i\omega\omega_i}$$

Here is the corresponding Matlab code:

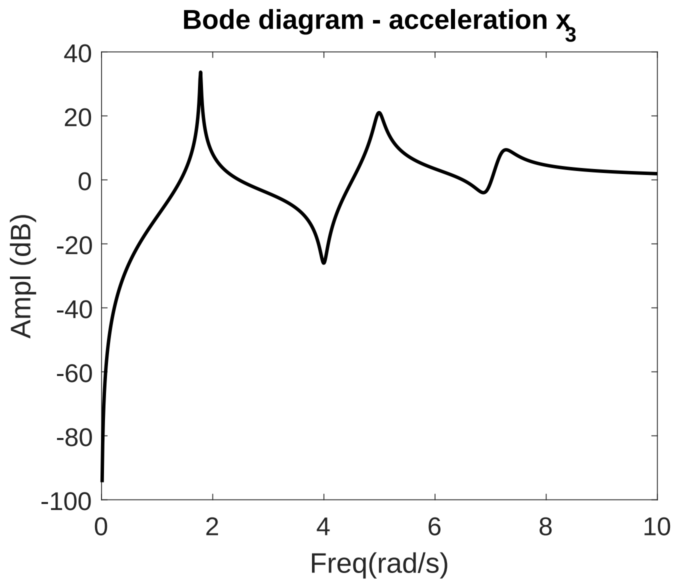
```
%% Projection in the modal basis
mui=diag(V'*M*V); wi=sqrt(diag(V'*K*V))./mui;
xi=(1./(2*mui.*wi)).*diag(V'*C*V);
wd=wi.*(sqrt(1-xi.^2));
F=V'*[0;0;1];

%% Impulse response
t=linspace(0,100,1000);
u=zeros(1,length(t));
for i=1:3
    u=u+V(3,i)*(F(i)*exp(-xi(i)*wi(i)*t)/(mui(i)*wd(i))).*sin(wd(i)*t);
end
figure; set(gca,'Fontsize',15)
plot(t,u); xlabel('time (sec)'); ylabel('Amplitude (m)')
title('impulse response - x_3')
```




```
%% Bode diagram
w=linspace(0,10,1000);
U=zeros(1,length(w));

for i=1:3
 U=U+ (F(i)*V(3,i))./(wi(i)^2 - w.^2 + 2*j*xi(i)*wi(i)*w);
end
figure; set(gca,'Fontsize',15)
plot(w,20*log10(abs(U)),'k','linewidth',2); hold on
xlabel('Freq(rad/s)'); ylabel('Ampl (dB)')

xi(2)=xi(2)*5;
U2=zeros(1,length(w));
for i=1:3
 U2=U2+ (F(i)*V(3,i))./(wi(i)^2 - w.^2 + 2*j*xi(i)*wi(i)*w);
end
plot(w,20*log10(abs(U2)),'r','linewidth',2)
legend('initial','5 times damp mode 2')
title('Bode diagram - displacement x_3')
```



```
% Acceleration
A=(-w.^2).*U;
figure; set(gca,'Fontsize',15)
plot(w,20*log10(abs(A)),'k','linewidth',2);
xlabel('Freq(rad/s)'); ylabel('Ampl (dB)')
title('Bode diagram - acceleration x_3')
```

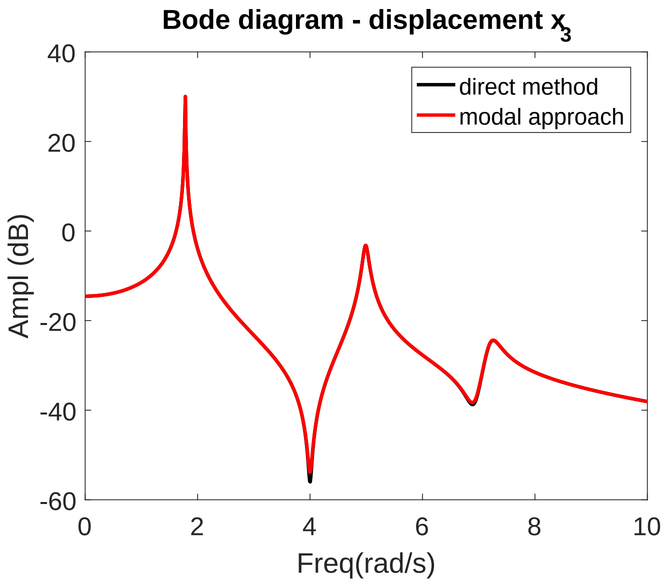


The modal damping hypothesis is valid because the damping matrix is proportional to the stiffness matrix. When multiplying by 5 the damping factor of mode 2, the frequency response function at x_3 is only affected around the second natural frequency of the system.

D) Consider the case when c_1=0. Is the modal damping hypothesis still verified ? Draw the Bode diagram for x_3 using the full system of equations (solve frequency by frequency). Compare with the modal approach in which the coupling is neglected. Comment

ANSWER:

```
c1=0;
C=[c1+c -c 0 ; -c 2*c -c ; 0 -c c];

% Direct reference method
U=zeros(3,length(w));
for i=1:length(w)
    U(:,i)=(K-w(i)^2*M + j*w(i)*C)\[0 0 1]';
end
figure; set(gca,'Fontsize',15)
plot(w,20*log10(abs(U(3,:))),'k','linewidth',2); hold on
xlabel('Freq(rad/s)'); ylabel('Ampl (dB)')

% Modal damping hypothesis
xi=(1./(2*mui.*wi)).*diag(V'*C*V);
U2=zeros(1,length(w));
for i=1:3
 U2=U2+ (F(i)*V(3,i))./(wi(i)^2 - w.^2 + 2*j*xi(i)*wi(i)*w);
end
plot(w,20*log10(abs(U2)),'r','linewidth',2)
legend('direct method','modal approach')
title('Bode diagram - displacement x_3')
```



The damping matrix is not proportional to the stiffness matrix anymore. Nevertheless, as the damping is quite small (only a few percents for each mode, see values of ξ_i in Matlab), the approximation is very good and there is almost no difference between the direct exact method and the decoupled modal approach.

Note: Use the `eig` function to compute the eigenfrequencies and mode shapes.